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Non-Renormalisability of Electromagnetic Self-Force

A Classical Picture

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Abstract We consider a rigid charge dumbbell, i.e., two point charges held some distance apart by an unspecified binding force, such that, when one charge follows an arbitrary timelike worldline in Minkowski spacetime, the motion of the other is completely specified by the condition that the axis of the system is Fermi–Walker transported along that worldline. The electromagnetic force the system exerts on itself under such conditions is calculated to leading order and found as usual to go as the reciprocal of the distance between the charges. However, it is shown that this term, which diverges in the point particle limit, is not proportional to the four-acceleration of the system except in special cases, whence the relativistic extension of Newton's second law cannot be renormalised in this limit. It is shown how this problem is resolved when the charge dumbbell is replaced by a spherically symmetric charge distribution. It is also shown how it can be resolved for the charge dumbbell by considering the stabilising forces (Poincaré stresses) that hold the system together, and which must themselves engender a self-force when the system is constrained to move rigidly. An alternative definition of the self-force is also considered and shown to lead to a consistent equation of motion for the dumbbell, and also for general charge distributions.

Keywords Maxwell electromagnetism, electromagnetic self-force, classical mass renormalisation, Poincaré stresses

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1 Introduction

In Sect. 2, we obtain a formula for the EM fields very close to the worldline of a point charge with arbitrary motion in Minkowski spacetime, following the analysis by Dirac [2]. In this paper, Dirac showed how to renormalise the mass of a classical point electron by assuming it was a point particle and considering conservation of energy and momentum within a tube containing the worldline, an analysis which led him to the notorious Lorentz–Dirac equation for the motion of a point charge. This equation is based in part on an (infinite) adjustment of the electron mass to cater for the (infinite) energy in the EM fields close to the charge [5, Chap. 11].

In this paper, the aim is to consider the electron as a spatially extended entity, making the very simple charge dumbbell model of two point charges held some distance apart by unspecified binding forces and calculating the EM force the system exerts upon itself when accelerated in some arbitrary way through Minkowski spacetime. Electromagnetic self-forces of this kind are discussed at length in [7]. The leading order term is not Coulomb, i.e., going as d^{-2}, where d is the separation of the point charges, but in fact goes as d^{-1}. In the point particle limit where one allows d to go to zero, this introduces an infinite self-force.

However, in some cases, the offending term can be absorbed into the mass times acceleration term of Newton's second law. This requires it to be proportional to the acceleration vector (four-vector or three-vector, depending on whether one considers a fully relativistic analysis or not). The mass of the charge gets adjusted by an amount that goes to infinity in the point particle limit, but this does not have physically measurable consequences if one considers the resulting m to be the physically measured mass. The fact that one has to go through such a procedure is just an inconvenient truth about the point particle approximation, and one which carries over to quantum field theory.

In [7], it was conjectured that the divergent leading order term in the EM self-force would always be aligned with the acceleration vector, so that mass renormalisation would always be possible, and that this might be the case precisely because Maxwell's theory is a gauge theory, since we know that all gauge theories are fully renormalisable in quantum field theory. In the present paper it is shown that, regarding the mass of the charge dumbbell system, classical electromagnetism will in fact only be renormalisable in some special cases, precisely those considered in [7], but that the situation is saved for spherically symmetric charge distributions.

This seems to suggest that the point particle limit may only be valid in such a case, viz., where the point particle is considered to be spherically symmetric, but this is not so. It will also be shown that, when we take into account the stabilising forces needed to hold the system together, often known as Poincaré stresses, the mass will always be renormalisable in the point particle limit. This is because, at least in the case where this system is stabilised in such a way as to be able to move rigidly under the arbitrary acceleration, the stabilising forces have to engender their own self-force, and it is one which precisely cancels the part of the EM self-force that is not aligned with the acceleration. This solution to the problem means that such a system is always renormalisable in the point particle limit, even if the EM self-force alone is not.

Some assumptions have to be made about the charge dumbbell in order to make any model of its motion under acceleration (see Sect. 3). If the two charges are labelled A and B, we attribute an arbitrary timelike worldline $x_A(\tau_A)$ to A and specify the motion of $x_B(\tau_B)$ of B by assuming a relativistic version of rigidity [1,6], and further assuming that the system is as rotationless as possible, i.e., that the system axis is Fermi–Walker transported along the worldline x_A. Calculations relaxing the last assumption might also be interesting, but more difficult.

As mentioned above, it is shown that the leading order term in the EM self-force is not generally parallel to the four-acceleration $\ddot{x}_A$ of particle A (see Sect. 4), whence the relativistic version of Newton's dynamical equation cannot generally be renormalised to cater for this in a point particle limit. We consider two simple cases where the EM self-force is in fact aligned with the acceleration, viz., linear acceleration normal to the dumbbell axis and linear acceleration along the dumbbell axis. Then the very same formulas are used to show in Sect. 5 that, if the system has spherical symmetry, e.g., if one considers a rigid spherical shell of charge, the EM self-force is proportional to the four-acceleration, and the mass of the shell can be renormalised, a well known result.

In Sect. 6, we consider the non-relativistic limit of the EM self-force and note that the offending term in the EM self-force does not become negligible, whence this is not an entirely relativistic problem.

In Sect. 7, we show how, in the case where this system is stabilised in such a way as to be able to move rigidly under the arbitrary acceleration, the stabilising Poincaré stresses have to engender their own self-force, and it is one which precisely cancels the part of the EM self-force that is not aligned with the acceleration. For this, we use a 'pressure' model of the Poincaré stresses due to Steane [10].

In Sect. 8, we introduce a recent idea in [9] which suggests an alternative way to ensure renormalisability of the equation of motion in the point particle limit. We examine this further in Sects. 9–11 in the light of observations made by Steane [10] regarding the problem that there is generally no covariant definition for the self-force (Sect. 9). Two definitions of the EM self-force and their possible physical justifications are described in Sect. 10. The more general issue of defining physical quantities 'for' accelerating observers is discussed in Sect. 11.

Appendix A presents the method of Ori and Rosenthal in detail, following [9]. These authors have shown that one may use an alternative definition of the EM self-force to obtain an equation of motion for the charge dumbbell in the point particle limit that is entirely consistent with the usual definition. Here we point out with Steane [10] that the self-force due to internal forces (Poincaré stresses) must be zero when we use this same altered definition for sums of forces acting at different spacetime events, and we show by generalising a method due to Steane [10] that this will indeed be the case, provided that the motion of the dumbbell is rigid. We also suggest that this whole method depends heavily on the rigidity of the motion and would not be generalisable to 'particles' whose internal structure involves a dynamic equilibrium, such as the proton.

We argue against Steane's proposal [10] that Ori and Rosenthal's method [9] is merely a natural choice for a certain kind of observer, namely one comoving with the dumbbell. What matters is the equation of motion of the system and its resulting worldtube, which is a coordinate-independent object, and in the present case, it is shown that the new method leads to exactly the same worldtube in the point particle limit, and hence constitutes a consistent reanalysis of this problem.

Appendix B provides plausible arguments for extending the dumbbell analysis of Sect. 7 to arbitrary continuous charge distributions in arbitrary rigid motion, using the standard way of summing forces acting at different spacetime events. However, we note that the 'pressure' model of the Poincaré stresses becomes somewhat strained, in the sense that it would be hard to imagine its physical implementation. Here it would be useful to find a more elegant formulation of these effects.

Appendix C discusses the role of Steane's 'hidden momentum' [10] in the case of the dumbbell with arbitrary rigid acceleration using the standard way of summing forces acting at different spacetime events and asks whether this gives a more insightful formulation.

2 EM Fields Close to the Worldline of a Point Charge

Here we follow the analysis in [2]. The Liénard–Wiechert retarded potential for the EM fields produced by a point charge e following a worldline $z(s)$ in Minkowski spacetime, with metric diag$(1,-1,-1,-1)$, is given as a function of the field point x by [7, Chap. 2]

$$A^\mu(x) = \left.\frac{e\dot{z}^\mu}{\dot{z}\cdot(x-z)}\right|_{s_+} ,$$

where the dot over a symbol denotes differentiation with respect to the proper time s along the worldline, and everything is evaluated at the unique retarded proper time s_+ for the given field point, i.e., the value of s such that

$$\big[x - z(s_+)\big]^2 = 0 , \qquad z^0(s_+) < x^0 . \tag{1}$$

Now for any two continuous functions $f(s)$ and $g(s)$ of s with $\dot{g}(s) > 0$, we have the following result from distribution theory:

$$\int f(s)\delta\big(g(s)\big)\mathrm{d}s = \int \frac{f(s)}{\dot{g}(s)}\delta\big(g(s)\big)\mathrm{d}g(s) = \frac{f(s)}{\dot{g}(s)} ,$$

evaluating the result at any value of s in the range of integration that satisfies $g(s) = 0$. By this ploy, we establish the more convenient result

$$A^\mu(x) = 2e\int \dot{z}^\mu(s)\delta\Big(\big[x - z(s)\big]^2\Big)\mathrm{d}s ,$$

integrating s from $-\infty$ to any value between the retarded and advanced proper times for the given field point x. One can relax the restriction on the range of integration by introducing a step function $\theta\big(x^0 - z^0(s)\big)$ into the integrand. But using the form given here, we have

$$\begin{aligned}
\frac{\partial A^\mu}{\partial x_\nu} &= 4e\int \dot{z}^\mu(x^\nu - z^\nu)\delta'\Big(\big[x - z(s)\big]^2\Big)\mathrm{d}s \\
&= -2e\int \frac{\dot{z}^\mu(x^\nu - z^\nu)}{\dot{z}_\lambda(x^\lambda - z^\lambda)}\frac{\mathrm{d}}{\mathrm{d}s}\delta\Big(\big[x - z(s)\big]^2\Big)\mathrm{d}s \\
&= 2e\int \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\dot{z}^\mu(x^\nu - z^\nu)}{\dot{z}_\lambda(x^\lambda - z^\lambda)}\right]\delta\Big(\big[x - z(s)\big]^2\Big)\mathrm{d}s ,
\end{aligned}$$

whence it follows that

$$\begin{aligned}
F^{\mu\nu}(x) &:= \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \\
&= -2e\int \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\dot{z}^\mu(x^\nu - z^\nu) - \dot{z}^\nu(x^\mu - z^\mu)}{\dot{z}_\lambda(x^\lambda - z^\lambda)}\right]\delta\Big(\big[x - z(s)\big]^2\Big)\mathrm{d}s ,
\end{aligned}$$

and finally,

$$F^{\mu\nu}(x) = -\frac{e}{\dot{z}\cdot(x-z)}\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\dot{z}^\mu(x^\nu - z^\nu) - \dot{z}^\nu(x^\mu - z^\mu)}{\dot{z}\cdot(x-z)}\right]\Bigg|_{s_+} , \qquad (2)$$

where everything in the last expression is evaluated at the retarded proper time s_+.

We now choose a field point x very close to the worldline. If it is close enough, there will be a unique proper time s_0 such that x is simultaneous with the charge at $z(s_0)$ for an observer instantaneously comoving with the charge. So s_0 is defined to be the unique proper time such that

$$\big[x^\mu - z^\mu(s)\big]\cdot v(s) = 0 , \qquad v(s) := \dot{z}(s) .$$

We define the spacelike vector from $z(s_0)$ to the field point, viz.,

$$\gamma^\mu := x^\mu - z^\mu(s_0) . \qquad (3)$$

Hence, we have

$$\gamma\cdot v(s_0) = 0 . \qquad (4)$$

By hypothesis, the γ^μ will be very small.

To apply (2), we need to consider the retarded time for the given field point x. We know there will be some small $\sigma \in \mathbb{R}^+$ such that the retarded time is $s_0 - \sigma$. We expect σ to be of the same order of magnitude as the γ^μ. The aim now is to expand the right-hand side of (2) in powers of σ. Of course, the leading order term will go as σ^{-2}. Following Dirac, we retain terms up to $O(\sigma^0)$, even though we shall only require those up to $O(\sigma^{-1})$ for present purposes.

We begin with the Taylor expansions

$$x^\mu - z^\mu(s_0 - \sigma) = \gamma^\mu + \sigma v^\mu - \frac{1}{2}\sigma^2\dot{v}^\mu + \frac{1}{6}\sigma^3\ddot{v}^\mu + O(\sigma^4) , \qquad (5)$$

$$\dot{z}^\mu(s_0 - \sigma) = v^\mu - \sigma\dot{v}^\mu + \frac{1}{2}\sigma^2\ddot{v}^\mu + O(\sigma^3) ,$$

where v, $\dot{v}$, and $\ddot{v}$ on the right-hand side are all evaluated at $s = s_0$, and it should be remembered that $\gamma \sim \sigma$. Applying the trivial results

$$v^2 = 1 , \qquad v\cdot\dot{v} = 0 , \qquad v\cdot\ddot{v} + \dot{v}^2 = 0 ,$$

and also (4), we obtain

$$\dot{z}\cdot(x-z) = \sigma - \sigma(\gamma\cdot\dot{v}) + \frac{1}{2}\sigma^2(\gamma\cdot\ddot{v}) - \frac{1}{6}\sigma^3\dot{v}^2 + O(\sigma^4) ,$$

and consequently,

$$\big[\dot{z}\cdot(x-z)\big]^{-1} = \sigma^{-1}\big[1 - (\gamma\cdot\dot{v})\big]^{-1}\left[1 - \frac{1}{2}\sigma(\gamma\cdot\ddot{v}) + \frac{1}{6}\sigma^2\dot{v}^2 + O(\sigma^3)\right] .$$

For the moment, we refrain from expanding out $\big[1-(\gamma\cdot\dot v)\big]^{-1}$. We also have

$$\dot z^\mu(x^\nu - z^\nu) - \dot z^\nu(x^\mu - z^\mu)$$
$$= v^\mu\gamma^\nu - \sigma\dot v^\mu\gamma^\nu - \frac{1}{2}\sigma^2\dot v^\mu v^\nu + \frac{1}{2}\sigma^2\ddot v^\mu\gamma^\nu + \frac{1}{3}\sigma^3\ddot v^\mu v^\nu - (\mu\longleftrightarrow\nu)\ ,$$

where once again the quantities v, $\dot v$, and $\ddot v$ on the right-hand side are evaluated at $s=s_0$. We now have

$$\frac{\dot z^\mu(x^\nu - z^\nu) - \dot z^\nu(x^\mu - z^\mu)}{\dot z\cdot(x-z)}$$
$$= \big[1-(\gamma\cdot\dot v)\big]^{-1}\Bigg[\sigma^{-1}v^\mu\gamma^\nu - \dot v^\mu\gamma^\nu - \frac{1}{2}\sigma\dot v^\mu v^\nu - \frac{1}{2}(\gamma\cdot\ddot v)v^\mu\gamma^\nu$$
$$+\frac{1}{6}\sigma\dot v^2 v^\mu\gamma^\nu + \frac{1}{2}\sigma\ddot v^\mu\gamma^\nu + \frac{1}{3}\sigma^2\ddot v^\mu v^\nu - (\mu\longleftrightarrow\nu)\Bigg]\ .$$

In the formula for the EM fields, this has to be differentiated with respect to s, and this can be done by differentiating with respect to σ and changing the sign, because we have not yet applied the condition that fixes σ. We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\dot z^\mu(x^\nu - z^\nu) - \dot z^\nu(x^\mu - z^\mu)}{\dot z\cdot(x-z)}\right]$$
$$= -\big[1-(\gamma\cdot\dot v)\big]^{-1}\Bigg[-\sigma^{-2}v^\mu\gamma^\nu - \frac{1}{2}\dot v^\mu v^\nu + \frac{1}{6}\dot v^2 v^\mu\gamma^\nu$$
$$+\frac{1}{2}\ddot v^\mu\gamma^\nu + \frac{2}{3}\sigma\ddot v^\mu v^\nu - (\mu\longleftrightarrow\nu)\Bigg]\ ,$$

which leads us to an expansion for the EM fields in powers of σ :

$$F^{\mu\nu} = e\big[1-(\gamma\cdot\dot v)\big]^{-2}\Bigg[-\sigma^{-3}v^\mu\gamma^\nu - \frac{1}{2}\sigma^{-1}\dot v^\mu v^\nu + \frac{1}{2}\sigma^{-2}(\gamma\cdot\ddot v)v^\mu\gamma^\nu$$
$$+\frac{1}{2}\sigma^{-1}\ddot v^\mu\gamma^\nu + \frac{2}{3}\ddot v^\mu v^\nu - (\mu\longleftrightarrow\nu)\Bigg]\ .$$

We now determine σ itself in powers of the small quantity $\epsilon\in\mathbb{R}^+$ defined by

$$\gamma^2 = -\epsilon^2\ . \tag{6}$$

Now σ is determined by (1), into which we insert the approximation (5) to obtain

$$\gamma^2 + \sigma^2 - \sigma^2(\gamma\cdot\dot v) + \frac{1}{3}\sigma^3(\gamma\cdot\ddot v) - \frac{1}{12}\sigma^4\dot v^2 = 0\ ,$$

which is correct to $O(\sigma^5)$. But $\sigma = \epsilon$ to first order, so this can be written

$$-\epsilon^2 + \sigma^2 - \sigma^2(\gamma\cdot\dot{v}) + \frac{1}{3}\epsilon^3(\gamma\cdot\ddot{v}) - \frac{1}{12}\epsilon^4\dot{v}^2 = 0\ .$$

Hence

$$\sigma^2 = \big[1-(\gamma\cdot\dot{v})\big]^{-1}\left[\epsilon^2 - \frac{1}{3}\epsilon^3(\gamma\cdot\ddot{v}) + \frac{1}{12}\epsilon^4\dot{v}^2\right]\ ,$$

and finally,

$$\sigma = \epsilon\big[1-(\gamma\cdot\dot{v})\big]^{-1/2}\left[1 - \frac{1}{6}\epsilon(\gamma\cdot\ddot{v}) + \frac{1}{24}\epsilon^2\dot{v}^2\right]\ .$$

Inserting this into the above expression for the EM fields, we have

$$F^{\mu\nu} = e\big[1-(\gamma\cdot\dot{v})\big]^{-1/2}\Bigg\{ -\epsilon^{-3}v^\mu\gamma^\nu - \frac{1}{2}\epsilon^{-1}\dot{v}^\mu v^\nu\big[1+(\gamma\cdot\dot{v})\big] + \frac{1}{8}\epsilon^{-1}\dot{v}^2 v^\mu\gamma^\nu + \frac{1}{2}\epsilon^{-1}\ddot{v}^\mu\gamma^\nu + \frac{2}{3}\ddot{v}^\mu v^\nu - (\mu\longleftrightarrow\nu)\Bigg\}\ . \tag{7}$$

This is Dirac's result [2]. It is accurate to $O(\epsilon^0)$ and could be used to calculate EM self-forces to this order in the linear dimensions ϵ of a small charge distribution. Terms of $O(\epsilon^0)$ explain EM radiation by such a system when it is accelerated [7].

In the present paper, we shall only require terms to $O(\epsilon^{-1})$, which diverge as the linear dimensions of our charge distribution tend to zero, i.e., when $\epsilon \to 0$. These are the terms that require renormalisation, even in the classical context, if we insist on point particles. To this order, we have

$$F^{\mu\nu} = e\big[1-(\gamma\cdot\dot{v})\big]^{-1/2}\left[-\epsilon^{-3}v^\mu\gamma^\nu - \frac{1}{2}\epsilon^{-1}\dot{v}^\mu v^\nu - (\mu\longleftrightarrow\nu)\right]\ .$$

We can also expand out the factor

$$\big[1-(\gamma\cdot\dot{v})\big]^{-1/2} = 1 + \frac{1}{2}(\gamma\cdot\dot{v}) + O(\epsilon^2)\ ,$$

whence finally, defining the unit spacelike vector $u := \gamma/\epsilon$,

$$F^{\mu\nu} = e\left[\frac{u^\mu v^\nu - v^\mu u^\nu}{\epsilon^2} + \frac{v^\mu\dot{v}^\nu - \dot{v}^\mu v^\nu + (u^\mu v^\nu - v^\mu u^\nu)(u\cdot\dot{v})}{2\epsilon} + O(\epsilon^0)\right]\ . \tag{8}$$

This is the result we shall apply below.

3 Charge Dumbbell and Rigidity Assumption

We wish to consider the simplest possible spatially extended charge distribution, namely two point charges e_A and e_B held some distance d apart by an unspecified binding force. When A has an arbitrary timelike worldline $x_A(\tau_A)$ in Minkowski spacetime, we know from Sect. 2 the EM field $F_A^{\mu\nu}(B)$ it will produce at the nearby point B, provided we also know the worldline $x_B(\tau_B)$ of B. We can then calculate the EM four-force of A on B from the standard result

$$F^\mu(A \text{ on } B) = e_B F_A^{\mu\nu}(B) v_\nu^B \ , \tag{9}$$

where v_B is the four-velocity of B. We can then find the four-force of B on A in a similar way and we simply add the two together, even though they act at different points of the charge distribution. This gives a total EM force of the system on itself for arbitrary motion. We shall look critically at this sum of four-forces in Sect. 9.

We call this the EM self-force F_{self}. The idea then is to expand F_{self} as a power series in the length d of the system. We expect the Coulomb terms going as d^{-2} to cancel out, but we also expect terms going as d^{-1}, d^0, and so on, to remain. In the point particle limit $d \to 0$, we need to be able to absorb the divergent term into the mass times acceleration component of the relativistic version of Newton's second law, viz.,

$$F = \frac{\mathrm{d}P}{\mathrm{d}\tau} = m\frac{\mathrm{d}a}{\mathrm{d}\tau} \ , \qquad P := mv \ ,$$

where m is rest mass, v is four-velocity, F is four-force, a is four-acceleration, and τ is proper time. To do this, it must clearly be proportional to the four-acceleration of the system. If it is not, this ploy, known as renormalisation, will not work.

But how do we formulate the motions of A and B? We would like to attribute an arbitrary worldline $x_A(\tau_A)$ to A. But then this constrains the worldline of B, which is supposed to be a distance d from it. On the other hand, whenever the system has a component of its motion along its own axis, we expect some degree of relativistic contraction, and this contraction will depend on the equilibrium between the unspecified binding forces and the EM forces, whereas we wish to do the calculation without going into too much detail about the binding forces. So what would be a good rule for constraining the worldline of B?

A related problem is that we will specify the two worldlines as spacetime functions of proper time, and the proper times τ_A and τ_B of the two charges will depend on the worldlines, whereas our self-force calculation must work out the four-force of A on B and then the four-force of B on A at spacetime events that are somehow physically relevant when we evaluate their sum, and that is not necessarily at the same coordinate time in whatever inertial frame we have selected at the outset (see Sect. 9).

One solution to these conundrums is to assume that the charge dumbbell is rigid, in a well defined relativistic sense of the word [1,6], and also that it is non-rotating, at least as far as this is possible. The purpose of this section is to explain briefly how this works. In-depth discussions of rigid motion can be found in [6] or [8].

We consider an orthonormal triad $n_i^\mu(\tau_A)$ $i=1,2,3$, of spacelike vectors along x_A, orthogonal to the worldline at each value of τ_A, whence

$$n_i \cdot n_j = -\delta_{ij}\ , \qquad n_i \cdot v_A = 0\ , \qquad v_A^2 = 1\ ,$$

setting $c=1$. We assume that the worldline of B can be given by

$$x_B^\mu(\tau_B) = x_A^\mu\big(\tau_A(\tau_B)\big) + \xi^i n_i^\mu\big(\tau_A(\tau_B)\big)\ , \tag{10}$$

for some function $\tau_A(\tau_B)$ to be determined, where ξ^i are just three fixed numbers. Naturally, $\tau_A(\tau_B)$ will depend on these three numbers. It is clear from the form of (10) that, given any value τ_B of the proper time of B, the function $\tau_A(\tau_B)$ delivers the unique proper time of A such that $x_B^\mu(\tau_B)$ is simultaneous with the event $x_A^\mu\big(\tau_A(\tau_B)\big)$ in the instantaneously comoving inertial frame ICIF_A of A at its proper time τ_A.

For the moment we have not completely specified the choice of spacelike triad $\{n_i\}_{i=1,2,3}$. It can rotate as it moves up the worldline x_A. However, given the triad and the three numbers $\{\xi_i\}_{i=1,2,3}$, the worldline x_B of B is fully determined. Differentiating (10) with respect to τ_B, we have

$$v_B^\mu(\tau_B) = \dot{x}_B^\mu(\tau_B) = \Big[v_A^\mu\big(\tau_A(\tau_B)\big) + \xi^i \dot{n}_i^\mu\big(\tau_A(\tau_B)\big)\Big]\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B}\ ,$$

where the dot over n_i denotes differentiation with respect to τ_A. For each value of τ_A, we can express the three four-vectors $\dot{n}_i$ in terms of the orthonormal basis $\{v_A, n_1, n_2, n_3\}$:

$$\dot{n}_i^\mu(\tau_A) = a_{0i}(\tau_A)v_A^\mu(\tau_A) + \Omega_{ij}(\tau_A)n_j^\mu(\tau_A)\ ,$$

for three functions $a_{0i}(\tau_A)$ and nine functions $\Omega_{ij}(\tau_A)$. It is easy to show that only three of the latter are independent since

$$\Omega_{ij}(\tau_A) = -\Omega_{ji}(\tau_A)\ , \qquad i,j \in \{1,2,3\}\ .$$

Furthermore, the three functions $a_{0i}(\tau_A)$ are given by

$$a_{0i}(\tau_A) = -n_i(\tau_A)\cdot \dot{v}_A(\tau_A)\ , \qquad i \in \{1,2,3\}\ , \tag{11}$$

whence they may be interpreted as the three components of the absolute acceleration of A at proper time τ_A, i.e., the three spatial components of the four-acceleration of A in its instantaneous rest frame ICIF_A (the temporal component being zero in that frame).

We now have

$$v_B^\mu(\tau_B) = \bigg\{\Big[1+\xi^i a_{0i}\big(\tau_A(\tau_B)\big)\Big]v_A^\mu\big(\tau_A(\tau_B)\big) + \xi^i\Omega_{ij}\big(\tau_A(\tau_B)\big)n_j^\mu\big(\tau_A(\tau_B)\big)\bigg\}\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B}.$$

Dropping the arguments of the functions, this implies that

$$1 = v_B^2 = \Big[\big(1+\xi^i a_{0i}\big)^2 - \xi^i\xi^j\Omega_{ik}\Omega_{jk}\Big]\left(\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B}\right)^2\ ,$$

whence

$$\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B} = \Big[\big(1+\xi^i a_{0i}\big)^2 - \xi^i\xi^j \Omega_{ik}\Omega_{jk}\Big]^{-1/2} \ ,$$

with the right-hand side a function of $\tau_A(\tau_B)$. The function $\tau_A(\tau_B)$ itself can then be found by integrating this along the worldline $x_B(\tau_B)$, with the boundary condition $\tau_A(0) = 0$, so that the proper times of A and B are synchronised when either is zero.

Now one case in which the system is said to be rigid in the relativistic sense is when $\Omega_{ij} = 0$, for all $i, j \in \{1, 2, 3\}$. We then have

$$v_B^\mu(\tau_B) = \Big[1 + \xi^i a_{0i}\big(\tau_A(\tau_B)\big)\Big] v_A^\mu\big(\tau_A(\tau_B)\big)\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B} \ .$$

with

$$\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B} = \big(1+\xi^i a_{0i}\big)^{-1} \ . \tag{12}$$

One very significant feature of this case is thus that

$$v_B^\mu(\tau_B) = v_A^\mu\big(\tau_A(\tau_B)\big) \ .$$

So for a rigid motion of our system, if we choose any event $x_B(\tau_B)$ on the worldline of B and find the unique event $x_A\big(\tau_A(\tau_B)\big)$ on the worldline of A for which an inertial observer instantaneously comoving with A considers B to be simultaneous, both charges have the same four-velocity. So an inertial observer moving instantaneously with B at the event $x_B(\tau_B)$ would also consider the event $x_A\big(\tau_A(\tau_B)\big)$ to be simultaneous.

This symmetry is going to be very useful below and is more or less the entire justification for the rather artificial rigidity assumption. Note that it is not at all obvious how such a situation might arise physically. It would have to result from the balance of forces between binding effects and EM effects within the charge dumbbell, not to mention the way the system is accelerated. Still, like many approximations in physics, it is justified by making some kind of analysis possible rather than by physical considerations!

Thinking back to the analysis in Sect. 2, for any choice of τ_B and considering $x_B(\tau_B)$ as a field point at which to evaluate the EM fields due to A, we have Dirac's spacelike four-vector of (3):

$$\gamma^\mu(\tau_B) = x_B^\mu(\tau_B) - x_A^\mu\big(\tau_A(\tau_B)\big) = \xi^i n_i^\mu\big(\tau_A(\tau_B)\big) \ , \tag{13}$$

with

$$\gamma^2 = -\boldsymbol{\xi}^2 := -(\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 \ .$$

This is of course constant, because we assumed at the outset that (10) would be possible for some constant choice of the ξ^i. Naturally, this is part of the rigidity assumption. It says that whenever an inertial observer instantaneously comoving with A looks at B, or vice versa, the other charge always seems to be the same distance away. So $|\boldsymbol{\xi}| =: d$ is the constant length of the charge dumbbell as judged by any inertial observer instantaneously comoving with either charge, and it will correspond to ϵ of (6) when we come to work out the EM forces of either charge on the other.

A word should be said about the assumption that $\Omega_{ij} = 0$ for all values of $i, j \in \{1, 2, 3\}$. These numbers constitute an antisymmetric matrix, and hence generate rotations. However, setting them equal to zero does not completely save our triad from rotation relative to space in some initially chosen inertial frame. In fact it amounts to saying that the triad n_i^μ is Fermi–Walker transported along the worldline of particle A. Let us see briefly what this means.

Recalling that $v_A(\tau_A)$ is the 4-velocity of the worldline of A, the equation for Fermi–Walker transport of a contravector M^μ along the worldline is

$$\dot{M}^\mu = -(M \cdot \dot{v}_A)v_A + (M \cdot v_A)\dot{v}_A \ . \tag{14}$$

This preserves inner products, i.e., if M and N are FW transported along the worldline, then $M \cdot N$ is constant along the worldline. Furthermore, the tangent vector v_A to the worldline is itself FW transported along the worldline, and if the worldline is a spacetime geodesic (a straight line in Minkowski coordinates), then FW transport is the same as parallel transport.

Now recall that the Ω_{ij} were defined by

$$\dot{n}_i^\mu = a_{0i} v_A^\mu + \Omega_{ij} n_j^\mu \ . \tag{15}$$

When $\Omega_{ij} = 0$, this becomes

$$\dot{n}_i^\mu = a_{0i} v_A^\mu \ . \tag{16}$$

This is indeed the Fermi–Walker transport equation for n_i^μ, found by inserting $M = n_i$ into (14), because we insist on $n_i \cdot v_A = 0$ and we have $a_{0i} = -n_i \cdot \dot{v}_A$.

In fact, the orientation in spacetime of the local rest frame triad n_i^μ cannot be kept constant along a worldline unless that worldline is straight (we are referring to flat spacetimes here). Under Fermi–Walker transport, however, the triad remains as constantly oriented, or as rotationless, as possible, in the following sense: at each instant of time τ_A, the triad is subjected to a pure Lorentz boost without rotation in the instantaneous hyperplane of simultaneity. (On a closed orbit, this process can still lead to spatial rotation of axes upon return to the same space coordinates, an effect known as Thomas precession.) For a general non-Fermi–Walker transported triad, the Ω_{ij} are the components of the angular velocity tensor that describes the instantaneous rate of rotation of the triad in the instantaneous hyperplane of simultaneity.

Of course, given any triad n_i^μ at one point on the worldline, it is always possible to Fermi–Walker transport it to other points by solving (14). We are then saying that, if the motion of B is given by

$$x_B^\mu(\tau_B) = x_A^\mu\big(\tau_A(\tau_B)\big) + \xi^i n_i^\mu\big(\tau_A(\tau_B)\big) \ , \tag{17}$$

where the ξ^i are fixed numbers, then the motion of the resulting dumbbell is rigid.

We need to check carefully that we do have all the necessary details of the above symmetry. Suppose therefore that we are trying to find the EM force of B on A for some choice τ_A. We shall need an orthogonal triad $\{n_i^B(\tau_B)\}$ along the worldline of B. But we can take the triad at $x_B(\tau_B)$ to be

$$n_i^B(\tau_B) := n_i\big(\tau_A(\tau_B)\big) \;, \tag{18}$$

where $\tau_A(\tau_B)$ is the function we discussed above. This automatically satisfies

$$n_i^B \cdot n_j^B = -\delta_{ij} \;, \qquad n_i^B \cdot v_B = 0 \;,$$

since $v_B(\tau_B) = v_A\big(\tau_A(\tau_B)\big)$. Interestingly, $\{n_i^B\}$ is also FW transported along x_B. This is shown by

$$\begin{aligned}\dot{n}_i^B &= \dot{n}_i \frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B} \\ &= \Big[-(n_i\cdot\dot{v}_A)v_A + (n_i\cdot v_A)\dot{v}_A\Big]\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B} \\ &= -(n_i^B\cdot\dot{v}_B)v_B + (n_i^B\cdot v_B)\dot{v}_B \;,\end{aligned}$$

which is the equation for FW transport of n_i^B along x_B, as required.

Effectively, we now have two functions $\tau_A(\tau_B)$ and $\tau_B(\tau_A)$. They are bijective and mutual inverses (see below for the proof). If we begin with an event $x_A(\tau_A)$ on x_A, the unique event $x_B(\tau_B)$ on x_B such that $x_A(\tau_A)$ is simultaneous with it in the inertial frame instantaneously comoving with four-velocity $v_B(\tau_B)$ is $x_B\big(\tau_B(\tau_A)\big)$, where the four-velocity is $v_B\big(\tau_B(\tau_A)\big)$. Now Dirac's spacelike vector $\gamma_B^\mu(\tau_A)$ from the unique $x_B\big(\tau_B(\tau_A)\big)$ to the chosen $x_A(\tau_A)$ is in this case

$$\gamma_B^\mu(\tau_A) = x_A^\mu(\tau_A) - x_B^\mu\big(\tau_B(\tau_A)\big) = -\gamma^\mu\big(\tau_B(\tau_A)\big) \;, \tag{19}$$

using (13). We express this in terms of the triad $n_i^B\big(\tau_B(\tau_A)\big) = n_i(\tau_A)$ along x_B. But we know that

$$x_B^\mu\big(\tau_B(\tau_A)\big) = x_A^\mu(\tau_A) + \xi^i n_i^\mu(\tau_A) \;, \tag{20}$$

so

$$\gamma_B^\mu(\tau_A) = -\xi^i n_i^\mu(\tau_A) = -\xi^i n_i^{B\mu}(\tau_B) \;,$$

and

$$\gamma_B^2 = -\boldsymbol{\xi}^2 = -d^2 \;.$$

So the numbers corresponding to the ξ^i are $-\xi^i$ when we approach the problem from this way around. Furthermore, the absolute acceleration of B is

$$\begin{aligned}a_{0i}^B(\tau_B) &= -n_i\big(\tau_A(\tau_B)\big)\cdot\dot{v}_B(\tau_B) \\ &= -n_i\cdot\frac{\mathrm{d}}{\mathrm{d}\tau_B}v_A\big(\tau_A(\tau_B)\big) \\ &= -n_i\cdot\frac{\mathrm{d}v_A}{\mathrm{d}\tau_A}\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B} \\ &= \frac{a_{0i}}{1+\xi^j a_{0j}} \;,\end{aligned} \tag{21}$$

using (12). Also by (12), we have

$$\begin{aligned}\frac{\mathrm{d}\tau_B}{\mathrm{d}\tau_A} &= \frac{1}{1-\xi^i a^B_{0i}} \\ &= \frac{1}{1-\xi^i \dfrac{a_{0i}}{1+\xi^j a_{0j}}} \\ &= 1+\xi^i a_{0i} = \left(\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B}\right)^{-1} ,\end{aligned} \tag{22}$$

proving the above claim that the functions $\tau_A(\tau_B)$ and $\tau_B(\tau_A)$ are mutual inverses.

4 Leading Order Term in EM Self-Force

We now wish to use (8) and (9) to calculate the EM force of A on B for some choice of the proper time τ_B :

$$F^\mu(A \text{ on } B) = e_B F_A^{\mu\nu}(B) v_\nu^B \ . \tag{23}$$

To this we wish to add the EM force of B on A for some choice of the proper time τ_A :

$$F^\mu(B \text{ on } A) = e_A F_B^{\mu\nu}(A) v_\nu^A \ .$$

This is where we use the symmetry established above for rigid motion: we work out $F^\mu(A \text{ on } B)$ and $F^\mu(B \text{ on } A)$ at corresponding values of τ_A and τ_B, viz., when $\tau_A = \tau_A(\tau_B)$ with the latter function determined by integrating (12). The point is that the two charges then have the same four-velocity, so we can find an inertial frame in which the two events $x_A(\tau_A)$ and $x_B(\tau_B)$ are simultaneous and the two charges instantaneously at rest. If we worked out the two EM forces in this inertial frame, we could add them to get the total EM self-force at this instant in this frame. Since $v_A(\tau_A)$ and $v_B(\tau_B)$ are also equal for this choice of τ_A and τ_B, we could then boost back to the original inertial frame and obtain

$$F^\mu_{\text{self}} = e_B F_A^{\mu\nu}(B) v_\nu^B + e_A F_B^{\mu\nu}(A) v_\nu^A \ . \tag{24}$$

Here F_{self} can be considered as a function of either τ_A or τ_B, where τ_A has to be given by the function $\tau_A(\tau_B)$ discussed above. The rigidity assumption is playing a key role in facilitating all this! We discuss this way of adding together four-forces acting at different spacetime events in more detail in Sect. 9.1.

Now by (8),

$$\begin{aligned}&F_A^{\mu\nu}(B) \\ &\quad = e_A \left[\frac{u^\mu v_A^\nu - v_A^\mu u^\nu}{d^2} + \frac{v_A^\mu \dot{v}_A^\nu - \dot{v}_A^\mu v_A^\nu + (u^\mu v_A^\nu - v_A^\mu u^\nu)(u\cdot \dot{v}_A)}{2d} + O(d^0) \right] ,\end{aligned}$$

where $u = \gamma/d$ is the unit four-vector from $x_A\big(\tau_A(\tau_B)\big)$ to $x_B(\tau_B)$ and v_A and $\dot{v}_A$ are both evaluated at $\tau_A(\tau_B)$. We thus calculate

$$F^\mu(A \text{ on } B) = e_B F_A^{\mu\nu}(B) v_\nu^B(\tau_B) = e_A e_B \left[\frac{u^\mu}{d^2} + \frac{-\dot{v}_A^\mu + (u \cdot \dot{v}_A) u^\mu}{2d} + O(d^0) \right] , \tag{25}$$

using the fact that $u \cdot v_B = 0$, $v_A \cdot v_B = 1$, and $\dot{v}_A \cdot v_B = 0$. The latter follows from

$$\dot{v}_B = \dot{v}_A \frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_B} = \frac{\dot{v}_A}{1 + \xi^i a_{0i}} ,$$

together with the fact that $\dot{v}_B \cdot v_B = 0$. Likewise,

$$F^\mu(B \text{ on } A) = e_A F_B^{\mu\nu}(A) v_\nu^A(\tau_A) = e_A e_B \left[\frac{u_B^\mu}{d^2} + \frac{-\dot{v}_B^\mu + (u_B \cdot \dot{v}_B) u_B^\mu}{2d} + O(d^0) \right] ,$$

where $u_B = \gamma_B/d$ is the unit four-vector from $x_B\big(\tau_B(\tau_A)\big)$ to $x_A(\tau_A)$ and v_B and $\dot{v}_B$ are both evaluated at $\tau_B(\tau_A)$. By (19), $u_B = -u$.

It is this last observation that shows why the Coulomb terms going as $O(d^{-2})$ cancel in the total self-force. There is just one more approximation to be made in determining the $O(d^{-1})$ term in the self-force:

$$\dot{v}_B = \frac{\dot{v}_A}{1 + \xi^i a_{0i}} = \dot{v}_A + O(d) , \tag{26}$$

since $|\boldsymbol{\xi}| = O(d)$. Hence, finally,

$$\boxed{F_{\text{self}} = \frac{e_A e_B}{d} \Big[(u \cdot \dot{v}_A) u - \dot{v}_A \Big] + O(d^0)} . \tag{27}$$

This shows that, contradicting the claim in [7], the EM self-force is not generally aligned with the four-acceleration of the system, taking the latter to be the four-acceleration of charge A here, at least not when this self-force is calculated in the way proposed here. If this is the right way to calculate the EM self-force that will appear in the dynamical equation for the system (providing its motion as a whole), it will not therefore be possible in general to renormalise the relativistic version of Newton's dynamical law in such a way as to cater for this effect.

In Sect. 7, we show how to save the day by taking into account the self-force due to the hitherto unconsidered binding effects within the system, viz., the Poincaré stresses. For this, we generalise an argument due to Steane [10]. In Appendix C, we show how to account for the unwanted term in (27) as driving hidden momentum through the dumbbell axis.

There are two simple cases where the EM self-force is aligned with the four-acceleration of the system, in fact precisely the two cases considered in [7]:

- The charge dumbbell moves along a straight line perpendicular to its axis.
- The charge dumbbell moves along a straight line parallel to its axis.

Transverse Linear Acceleration

Taking the motion along the x axis,

$$x_A = (x_A^0, x_A^1, 0, 0) , \qquad v_A = (\dot{x}_A^0, \dot{x}_A^1, 0, 0) ,$$

where x_A^0, x_A^1 are functions of the proper time τ_A and dots over symbols denote differentiation with respect to τ_A. Likewise,

$$x_B = (x_B^0, x_B^1, d, 0) , \qquad v_B = (\dot{x}_B^0, \dot{x}_B^1, 0, 0) ,$$

where $\tau_B = \tau_A$, $x_B^0(\tau_B) = x_A^0(\tau_A)$, and $x_B^1(\tau_B) = x_A^1(\tau_A)$. We thus also have $v_B = v_A$ and $\dot{v}_B = \dot{v}_A$. Furthermore,

$$\gamma = (0, 0, d, 0) , \qquad u = (0, 0, 1, 0) .$$

Hence, $u \cdot \dot{v}_A = 0$ and we obtain simply

$$F_{\text{self}} = -\frac{e_A e_B}{d} \dot{v}_A + O(d^0) , \tag{28}$$

agreeing with the result in [7].

Longitudinal Linear Acceleration

Once again, we take the motion to be along the x axis. In the local rest frame of A, $u = (0, 1, 0, 0)$, so in the fixed inertial frame relative to which A has four-velocity $v_A(\tau_A)$,

$$u = \big(\gamma(w)w, \gamma(w), 0, 0\big) , \qquad v_A = \big(\gamma(w), \gamma(w)w, 0, 0\big) ,$$

where w is the coordinate velocity of A in the x direction and

$$\gamma(w) := (1 - w^2)^{-1/2} .$$

A little calculation shows that

$$\dot{v}_A = \gamma(w)^4 a(w, 1, 0, 0) , \qquad a := \frac{\mathrm{d}w}{\mathrm{d}t} ,$$

where a is the coordinate acceleration of A in the x direction. We then have

$$u \cdot \dot{v}_A = -\gamma(w)^3 a ,$$

so that, using (27) and a little manipulation,

$$F_{\text{self}} = -\frac{2e_A e_B}{d} \dot{v}_A + O(d^0) . \tag{29}$$

Once again this agrees with the result in [7].

5 Spherical Symmetry

In the last section, we identified two special cases in which the general result (27) does in fact lead to a renormalisable mass in the limit as the system size tends to zero. But the general result itself contains a term proportional to the separation four-vector u between the two charges which means that the leading order term in the expansion of F_{self}, going as d^{-1}, is not proportional to the four-acceleration vector of the system, whence renormalisation will not generally be possible (but see Sect. 7).

One way round this problem is to hypothesise that any real particle comprising a spatially extended charge distribution is spherically symmetric. It is in fact a well known result, for example, that the leading order term in the EM self-force of a spherical charge shell will be aligned with the four-acceleration, and indeed we can prove this from (27). This result is not wholly surprising. What vectors are left for F_{self} to pick out when the system geometry itself does not specify any particular vector? There is of course the four-velocity, but it turns out that F_{self} picks out the four-acceleration and mass renormalisation is then possible.

We thus consider a rigid spherical charge shell of radius R whose center follows an arbitrary timelike worldline in such a way that any line segment between diametrically opposite charge elements on the surface is Fermi–Walker transported along that worldline, i.e., the sphere undergoes as little rotation as possible in the sense described in Sect. 3. Rigidity means here that there is always an inertial frame in which the whole charge shell is instantaneously at rest, and that the shell is always spherical in that frame [6].

We consider the charge shell in its instantaneous (inertial) rest frame. The surface charge density is $\rho = e/4\pi R^2$, assuming a total charge of e on the surface. We take surface charge elements in pairs $\mathrm{d}\sigma_1 = e\mathrm{d}\Omega_1/4\pi$ and $\mathrm{d}\sigma_2 = e\mathrm{d}\Omega_2/4\pi$, where $\mathrm{d}\Omega_1$ and $\mathrm{d}\Omega_2$ are the corresponding solid angle elements subtended at the center. For each pair of charge elements there is a unit separation four-vector $u = (0, \mathbf{n})$.

There is also an acceleration vector $\dot{v} = (0, \mathbf{a})$ which can be the four-acceleration of either of the charge elements, or better still of the sphere center, since we know that these four-acceleration vectors differ by at most a term of $O(R)$ [see (26), for example]. Of course, the unit three-vector $\mathbf{n}$ will vary as we change the pair of surface charge elements, but the three-vector $\mathbf{a}$ will not, because we carry out our calculation for some particular instantaneous snapshot of the charge shell.

By (27), the EM self-force on this pair of charge elements is

$$\delta F_{\mathrm{self}}(\Omega_1, \Omega_2) = -\frac{e^2 \mathrm{d}\Omega_1 \mathrm{d}\Omega_2}{d(\Omega_1, \Omega_2)(4\pi)^2}\Big[(\mathbf{n}\cdot\mathbf{a})\mathbf{n} + \mathbf{a}\Big] ,$$

where $d(\Omega_1, \Omega_2)$ is a distance function, specifying the separation of the two charge elements in this frame, and $\mathbf{n}$ is of course also a function of the pair (Ω_1, Ω_2). The total EM self-force is thus

$$F_{\mathrm{self}} = -\frac{e^2}{2(4\pi)^2}\iint \frac{(\mathbf{n}\cdot\mathbf{a})\mathbf{n} + \mathbf{a}}{d(\Omega_1, \Omega_2)} \mathrm{d}\Omega_1 \mathrm{d}\Omega_2 ,$$

with an extra factor of 1/2 to account for the fact that we count each pair of charge elements twice in the integral.

We now write $\mathbf{n} = \mathbf{n}_\parallel + \mathbf{n}_\perp$, where $\mathbf{n}_\parallel$ is the component of $\mathbf{n}$ parallel to $\mathbf{a}$ and $\mathbf{n}_\perp$ is its component perpendicular to $\mathbf{a}$. Then

$$(\mathbf{n}\cdot\mathbf{a})\mathbf{n} = (\mathbf{n}_\parallel\cdot\mathbf{a})\mathbf{n}_\parallel + (\mathbf{n}_\parallel\cdot\mathbf{a})\mathbf{n}_\perp \ .$$

However, the integral of the second term here is zero:

$$\iint \frac{(\mathbf{n}_\parallel\cdot\mathbf{a})\mathbf{n}_\perp}{d(\Omega_1,\Omega_2)}\mathrm{d}\Omega_1\mathrm{d}\Omega_2 = 0 \ .$$

This is because, for every pair of solid angle elements $\mathrm{d}\Omega_1$ and $\mathrm{d}\Omega_2$, there is another pair $\mathrm{d}\Omega_1'$ and $\mathrm{d}\Omega_2'$ obtained by rotation through 180° about $\mathbf{a}$, for which $\mathbf{n}_\parallel = \mathbf{n}_\parallel'$ but $\mathbf{n}_\perp = -\mathbf{n}_\perp'$. Contributions therefore cancel.

Hence,

$$F_{\mathrm{self}} = -\frac{e^2}{2(4\pi)^2}\iint \frac{(\mathbf{n}_\parallel\cdot\mathbf{a})\mathbf{n}_\parallel + \mathbf{a}}{d(\Omega_1,\Omega_2)}\mathrm{d}\Omega_1\mathrm{d}\Omega_2 \ .$$

Now $\mathbf{n}_\parallel\cdot\mathbf{a} = n_\parallel a$, the product of the lengths of the two vectors, and

$$(an_\parallel)\mathbf{n}_\parallel = \mathbf{a}(n_\parallel)^2 = \mathbf{a}(\mathbf{e}\cdot\mathbf{n})^2 \ ,$$

where $\mathbf{e}$ is a unit vector parallel to $\mathbf{a}$. We now have

$$F_{\mathrm{self}} = -\frac{e^2\mathbf{a}}{2(4\pi)^2}\iint \frac{1+(\mathbf{e}\cdot\mathbf{n})^2}{d(\Omega_1,\Omega_2)}\mathrm{d}\Omega_1\mathrm{d}\Omega_2 \ ,$$

Note that by spherical symmetry the integral here has to be independent of $\mathbf{e}$. If $\mathbf{e}$ points into the solid angle Ω, then

$$\begin{aligned}\iint \frac{(\mathbf{e}\cdot\mathbf{n})^2}{d(\Omega_1,\Omega_2)}\mathrm{d}\Omega_1\mathrm{d}\Omega_2 &= \frac{1}{4\pi}\int \mathrm{d}\Omega \iint \frac{(\mathbf{e}\cdot\mathbf{n})^2}{d(\Omega_1,\Omega_2)}\mathrm{d}\Omega_1\mathrm{d}\Omega_2 \\ &= \frac{1}{4\pi}\iint \frac{\mathrm{d}\Omega_1\mathrm{d}\Omega_2}{d(\Omega_1,\Omega_2)}\int (\mathbf{e}\cdot\mathbf{n})^2\mathrm{d}\Omega \ ,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{4\pi}\int (\mathbf{e}\cdot\mathbf{n})^2\mathrm{d}\Omega &= \frac{1}{4\pi}\int_0^\pi \cos^2\theta\sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\ &= \frac{1}{2}\int_0^\pi \cos^2\theta\sin\theta\mathrm{d}\theta = \frac{1}{3} \ .\end{aligned}$$

So finally,

$$F_{\mathrm{self}} = -\frac{2}{3}\frac{e^2}{(4\pi)^2}\mathbf{a}\iint \frac{\mathrm{d}\Omega_1\mathrm{d}\Omega_2}{d} = -\frac{4}{3}U\mathbf{a} \ , \tag{30}$$

where

$$U := \frac{e^2}{(4\pi)^2}\frac{1}{2}\iint \frac{\mathrm{d}\Omega_1\mathrm{d}\Omega_2}{d}$$

is the EM energy of the charge shell in this frame.

Equation (30) is a standard result [7]. It shows that the EM self-force in this case is proportional to the four-acceleration of the spatially extended charge distribution. The equation of motion of such a particle can thus be renormalised in the limit as the size of the charge distribution goes to zero.

6 Non-Relativistic Limit

In this case, the hyperplanes of simultaneity of either A or B are approximately hyperplanes of simultaneity for the global inertial frame chosen at the outset, i.e., hyperplanes of constant inertial time t. Furthermore, the proper times of A and B are both approximately equal to inertial time, i.e., $\tau_A(t) \approx t$ and $\tau_B(t) \approx t$. So

$$x_B(\tau_B) - x_A(\tau_A) \approx (0, \text{space vector from } A \text{ to } B \text{ at time } t) \;.$$

This means that $u^0 \approx 0$, and $\mathbf{u}$ is approximately a unit space vector in the direction from A to B lying in the hyperplane of constant t at each time t. The four-velocities of A and B are

$$v_A \approx (c, \mathrm{d}\mathbf{x}_A/\mathrm{d}t) \;, \qquad v_B \approx (c, \mathrm{d}\mathbf{x}_B/\mathrm{d}t) \;,$$

and the four-accelerations

$$\dot{v}_A \approx (0, \mathrm{d}^2\mathbf{x}_A/\mathrm{d}t^2) \;, \qquad \dot{v}_B \approx (0, \mathrm{d}^2\mathbf{x}_B/\mathrm{d}t^2) \;.$$

We thus find

$$u \cdot \dot{v}_A \approx -\mathbf{u} \cdot \frac{\mathrm{d}^2\mathbf{x}_A}{\mathrm{d}t^2} \;,$$

which is minus the scalar product of the unit three-vector joining A to B in the constant t hyperplane and the instantaneous three-acceleration of A at that inertial time t. When this is not zero, as is generally the case, the term $(u \cdot \dot{v}_A)u$ in the self-force (27) will not be zero, and nor will it generally be aligned with the acceleration of A, since it lies along the unit three-vector joining A to B in the constant t hyperplane.

7 Total Self-Force on the Charge Dumbbell

The contents of this section are inspired by an observation by Steane in [10]. We can see by looking at (28) and (29) that the self-force-derived masses are different in the two cases, and only the first agrees with the energy-derived mass of the fields. We also know from the discussion in [7, Chap. 11] that discrepancies between energy-derived and *momentum-derived* masses, are ultimately due to not taking into account binding forces (see the note on this on pp. 4–5 of [9]), and we know that momentum gets into the EM fields by doing work against the leading order term in the self-force [7, Chap. 3], whence the momentum-derived and self-force-derived masses of the fields are always equal.

Steane makes the connection explicit in his paper [10], showing that the binding forces, or Poincaré stresses, must themselves engender a self-force which precisely readjusts the total self-force in such a way that the mass renormalisation will be the same for both longitudinal and transverse motion of the dumbbell in the point particle limit. He does this for the case of uniform acceleration.

The aim here will be to adapt his argument to the present case of general rigid acceleration without rotation. The idea is to imagine that the two charges are joined by a massless rod containing a massless fluid which is nevertheless capable of exerting the necessary binding force in some mechanical way. We then solve the relativistic Navier–Stokes equation for the pressure in the fluid and compare the forces it exerts at the two ends of the rod. This is clearly very heuristic. A better picture suggested by Steane would perhaps be to consider hidden momentum due to energy transfer along the rod [10]. This is discussed for the present case in Appendix C.

In the instantaneous rest frame of the rod and for $c = 1$, the relativistic Navier–Stokes equation is

$$(\rho + p)\mathbf{a} = -\boldsymbol{\nabla} p \,,$$

where ρ is the density of the fluid, taken to be zero here, $\mathbf{a}$ is the spatial part of the four-acceleration of the fluid, and p is the pressure. Note how this equation already shows how pressure should be associated with inertial mass in the relativistic context, since it appears side by side with the mass density. We would thus like to solve

$$p\,\mathbf{a} = -\boldsymbol{\nabla} p \tag{31}$$

for p as a function of spacetime events within the rod. The first thing to do is thus to obtain the acceleration $\mathbf{a}$ as a function of these events.

We have already seen in (21) that A and B at either end of the rod have different accelerations, and that these accelerations are related in a rather simple way that depends on the fixed parameters ξ^i. But it is obvious how to express the acceleration as a function of a parameter s that runs from 0 to 1 as we move from A to B. Just as we had (10), viz.,

$$x_B^\mu\big(\tau_B(\tau_A)\big) = x_A^\mu(\tau_A) + \xi^i n_i^\mu(\tau_A) \,,$$

for each $s \in [0,1]$, we can write

$$x_s^\mu\big(\tau_s(\tau_A)\big) = x_A^\mu(\tau_A) + s\xi^i n_i^\mu(\tau_A) \,, \tag{32}$$

where the right-hand side is taken to define a worldline $x_s^\mu\big(\tau_s(\tau_A)\big)$ as proper time τ_A goes by at A and we parametrise it by its own proper time τ_s, which we can take to be a function of τ_A in this relation. When $s = 0$, we get $x_A^\mu(\tau_A)$, and when $s = 1$, we get $x_B^\mu(\tau_B)$.

It is not hard to show that each worldline $x_s^\mu(\tau_s)$ is the worldline of a material point of the rigid rod connecting the charges, i.e., of the rigidly moving fluid we have imagined to fill that rod. One has to show that, in any instantaneous rest frame of the rod, the point is always the same distance from A or B. But up to a sign, that distance, the proper distance for that hyperplane of simultaneity, is just the pseudolength of the spacelike vector

$$\gamma_s^\mu(\tau_s) := x_s^\mu(\tau_s) - x_A^\mu\big(\tau_A(\tau_s)\big) = s\xi^i n_i^\mu\big(\tau_A(\tau_s)\big) \,,$$

which is just

$$\gamma_s^2 = -s^2\boldsymbol{\xi}^2 := -s^2\Big[(\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2\Big] = -s^2 d^2 \,,$$

by analogy with (13). This is constant for fixed s.

So it is the four-acceleration of the worldline $x_s^\mu(\tau_s)$ that we are after for each value of s in the interval $[0,1]$. The space part of this in the instantaneous rest frame can be inserted in (31). This will effectively give $\mathbf{a}$ as a function of our parameter s, and we can then consider the component of the equation (31) along the dumbbell axis in the instantaneous rest frame. Strictly speaking, we should have p as a function of three space coordinates, but we are only interested in the way p varies with s, along the rod. Variations across the rod would have no noticeable effect because it is assumed to be very thin, i.e., there is no room for p to change much (even though it will change across the rod). After all, the self-force due to this pressure effect comes from the pressure difference between the two ends of the rod. It is also clear that this will act along the rod, just as we would hope if it is to cancel the unwanted term in the EM self-force.

For each $s \in [0,1]$, we can obtain the same kind of formulas for the motion of the corresponding point as we did for the end point B. For example, we have an invertible function $\tau_s(\tau_A)$ giving τ_s in terms of τ_A, and for the four-velocity $v_s := \dot{x}_s$, we have

$$v_s^\mu(\tau_s) = v_A^\mu\big(\tau_A(\tau_s)\big) \;, \qquad v_s^\mu\big(\tau_s(\tau_A)\big) = v_A^\mu(\tau_A) \;. \tag{33}$$

By analogy with (12), we have

$$\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_s} = \big(1 + s\xi^i a_{0i}\big)^{-1} \;, \tag{34}$$

where, as before, the three functions $a_{0i}(\tau_A)$ are given by

$$a_{0i}(\tau_A) = -n_i(\tau_A)\cdot \dot{v}_A(\tau_A) \;, \qquad i \in \{1,2,3\} \;,$$

whence they may be interpreted as the three components of the absolute acceleration of A at proper time τ_A, i.e., the three spatial components of the four-acceleration of A in its instantaneous rest frame ICIF_A (the temporal component being zero in that frame).

Then by analogy with (21), the absolute acceleration of x_s is

$$\begin{aligned} a_{0i}^s(\tau_s) &= -n_i\big(\tau_A(\tau_s)\big)\cdot \dot{v}_s(\tau_s) \\ &= -n_i \cdot \frac{\mathrm{d}}{\mathrm{d}\tau_s} v_s(\tau_s) \\ &= -n_i \cdot \frac{\mathrm{d}}{\mathrm{d}\tau_s} v_A\big(\tau_A(\tau_s)\big) \qquad \text{[by (33)]} \\ &= -n_i \cdot \frac{\mathrm{d}v_A}{\mathrm{d}\tau_A}\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_s} \\ &= \frac{a_{0i}}{1 + s\xi^j a_{0j}} \qquad \text{[by (34)]} \;, \end{aligned} \tag{35}$$

where we have used the fact that

$$\dot{v}_A(\tau_A) = a_{0i}(\tau_A) n_i(\tau_A) \;. \tag{36}$$

Of course, all this implies that, defining $n_i^s(\tau_s) := n_i\big(\tau_A(\tau_s)\big)$ along the lines of (18), we have

$$\begin{aligned}\dot{v}_s(\tau_s) &= a_{0i}^s(\tau_s)n_i^s(\tau_s)\\ &= \frac{a_{0i}\big(\tau_A(\tau_s)\big)}{1+s\xi^j a_{0j}}n_i\big(\tau_A(\tau_s)\big)\\ &= \frac{\dot{v}_A\big(\tau_A(\tau_s)\big)}{1+s\xi^j a_{0j}}\ .\end{aligned}$$

Anyway, we shall use (35) in the Navier–Stokes equation (31).

But first, we take the component of (31) along the rod axis in the instantaneous rest frame by taking the 3D spatial scalar product with the three-vector

$$\mathbf{d}(\tau_A) := \xi^1\mathbf{n}_1(\tau_A) + \xi^2\mathbf{n}_2(\tau_A) + \xi^3\mathbf{n}_3(\tau_A)\ .$$

Note that the three-vector part of (32) now reads

$$\mathbf{x}_s^\mu\big(\tau_s(\tau_A)\big) = \mathbf{x}_A^\mu(\tau_A) + s\xi^i\mathbf{n}_i^\mu(\tau_A) = \mathbf{x}_A^\mu(\tau_A) + s\mathbf{d}(\tau_A)\ . \tag{37}$$

Of course, in the instantaneous rest frame of the whole system at proper time τ_A for A, and we know that there is such a rest frame due to the rigidity of the motion,

$$\gamma^\mu(\tau_A) := \xi^i n_i^\mu(\tau_A) = (0,\mathbf{d}) = (0,\xi^1,\xi^2,\xi^3)\ ,$$

where the last expression gives the components of this vector in the tetrad basis, and also

$$\gamma_s^\mu(\tau_A) := s\xi^i n_i^\mu(\tau_A) = (0,s\mathbf{d}) = (0,s\xi^1,s\xi^2,s\xi^3)\ .$$

Also in this instantaneous rest frame, the four-accelerations of A and the point parametrised by s are

$$\dot{v}_A(\tau_A) = (0,a_{01},a_{02},a_{03}) = (0,\mathbf{a}_0)\ ,$$

$$\dot{v}_s(\tau_s) = \frac{(0,a_{01},a_{02},a_{03})}{1+s\xi^j a_{0j}} = \frac{(0,\mathbf{a}_0)}{1+s\xi^j a_{0j}} = \frac{(0,\mathbf{a}_0)}{1+s\mathbf{d}\cdot\mathbf{a}_0}\ ,$$

giving components relative to the tetrad basis at the relevant events. The scalar product in the denominator of the last expression is just the 3D scalar product of three-component vectors in the hyperplane of simultaneity of A at proper time τ_A.

If we think of the pressure as a function of s along the rod, viz.,

$$p\big(x(s),y(s),z(s)\big)\ ,$$

where x^i, $i=1,2,3$, are space coordinates in the ICIF, then we have

$$\mathbf{d}\cdot\boldsymbol{\nabla}p = d^i\frac{\partial p}{\partial x^i} = \frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\partial p}{\partial x^i} = \frac{\mathrm{d}p}{\mathrm{d}s}\ ,$$

using the fact that

$$\frac{\mathrm{d}x^i}{\mathrm{d}s} = d^i \,,$$

which follows from (37). The Navier–Stokes equation now reads

$$p\frac{\mathbf{d}\cdot\mathbf{a}_0}{1+s\mathbf{d}\cdot\mathbf{a}_0} = -\frac{\mathrm{d}p}{\mathrm{d}s} \,, \tag{38}$$

a 1D equation giving the pressure as a function of the rod parametrisation s. This implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln p = -\frac{\mathbf{d}\cdot\mathbf{a}_0}{1+s\mathbf{d}\cdot\mathbf{a}_0} = -\frac{\mathrm{d}}{\mathrm{d}s}\ln(1+s\mathbf{d}\cdot\mathbf{a}_0) \,,$$

which has solution

$$\boxed{p(s) = \frac{k}{1+s\mathbf{d}\cdot\mathbf{a}_0}} \,, \tag{39}$$

where k is a constant.

If we take this pressure to model the Poincaré stresses (PS) that prevent the dumbbell breaking up (or the dipole from collapsing), we can already see that it will itself produce a self-force that is directed along the rod in the ICIF. If $\mathcal{A}$ is the cross-sectional area of the rod, the PS self-force will be

$$p(1)\mathcal{A} - p(0)\mathcal{A} = \frac{k\mathcal{A}}{1+\mathbf{d}\cdot\mathbf{a}_0} - k\mathcal{A} = -\frac{k\mathcal{A}\mathbf{d}\cdot\mathbf{a}_0}{1+\mathbf{d}\cdot\mathbf{a}_0} \,.$$

This is promising for the present inquiry, but still not much help because we don't know k or $\mathcal{A}$.

Note that, although a simple pressure in the fluid would be enough to do the work required of a Poincaré stress and keep two unlike charges apart, for example, the self-force effect here is *not* simply due to a pressure force along the line joining A to B. Rather it is due to a pressure *difference* between the locations of A and B which would lead to a *net pressure self-force* on the combined system of the two charges. And this pressure difference along the imaginary column of fluid arises *because of* the constraint of rigid motion.

Here we follow the clever argument due to Steane [10]. Let $\mathbf{f}^A_{\mathrm{ext}}$ and $\mathbf{f}^B_{\mathrm{ext}}$ be the external three-forces acting on A and B to ensure the motion, as given in the ICIF. Then let $\mathbf{f}_A$ be the EM force on A due to B and $\mathbf{f}_B$ the EM force on B due to A. Let m be the observed mass of both A and B, and $\mathbf{a}_A$ and $\mathbf{a}_B$ the accelerations. We thus have the equations of motion of A and B:

$$\begin{aligned} \mathbf{f}^A_{\mathrm{ext}} + \mathbf{f}_A - p(0)\mathcal{A}\hat{\mathbf{d}} &= m\mathbf{a}_A \,, \\ \mathbf{f}^B_{\mathrm{ext}} + \mathbf{f}_B + p(1)\mathcal{A}\hat{\mathbf{d}} &= m\mathbf{a}_B \,, \end{aligned} \tag{40}$$

where $\hat{\mathbf{d}}$ is a unit vector along the rod. Taking components along the rod, we have

$$\mathbf{f}^A_{\mathrm{ext}}\cdot\hat{\mathbf{d}} + \mathbf{f}_A\cdot\hat{\mathbf{d}} - p(0)\mathcal{A} = m\mathbf{a}_A\cdot\hat{\mathbf{d}} = m\mathbf{a}_0\cdot\hat{\mathbf{d}} \,, \tag{41}$$

$$\mathbf{f}^B_{\mathrm{ext}}\cdot\hat{\mathbf{d}} + \mathbf{f}_B\cdot\hat{\mathbf{d}} + p(1)\mathcal{A} = m\mathbf{a}_B\cdot\hat{\mathbf{d}} = m\frac{\mathbf{a}_0\cdot\hat{\mathbf{d}}}{1+\mathbf{d}\cdot\mathbf{a}_0} \,. \tag{42}$$

We now eliminate the unknown product $k\mathcal{A}$ by writing

$$\frac{\mathbf{f}_{\rm ext}^A\cdot\hat{\mathbf{d}}+\mathbf{f}_A\cdot\hat{\mathbf{d}}-m\mathbf{a}_0\cdot\hat{\mathbf{d}}}{-\mathbf{f}_{\rm ext}^B\cdot\hat{\mathbf{d}}-\mathbf{f}_B\cdot\hat{\mathbf{d}}+m\dfrac{\mathbf{a}_0\cdot\hat{\mathbf{d}}}{1+\mathbf{d}\cdot\mathbf{a}_0}}=\frac{p(0)\mathcal{A}}{p(1)\mathcal{A}}=1+\mathbf{d}\cdot\mathbf{a}_0\ ,$$

using (39) in the last step. Hence,

$$\mathbf{f}_{\rm ext}^A\cdot\hat{\mathbf{d}}+\mathbf{f}_A\cdot\hat{\mathbf{d}}-m\mathbf{a}_0\cdot\hat{\mathbf{d}}=-\Big(\mathbf{f}_{\rm ext}^B\cdot\hat{\mathbf{d}}+\mathbf{f}_B\cdot\hat{\mathbf{d}}\Big)(1+\mathbf{d}\cdot\mathbf{a}_0)+m\mathbf{a}_0\cdot\hat{\mathbf{d}}\ .$$

Rearranging and writing the total external force on the system as

$$\mathbf{f}_{\rm ext}^{\rm tot}:=\mathbf{f}_{\rm ext}^A+\mathbf{f}_{\rm ext}^B\ ,$$

we obtain

$$\boxed{(\mathbf{f}_A+\mathbf{f}_B)\cdot\hat{\mathbf{d}}+\mathbf{f}_{\rm ext}^{\rm tot}\cdot\hat{\mathbf{d}}=2m\mathbf{a}_0\cdot\hat{\mathbf{d}}-\Big(\mathbf{f}_{\rm ext}^B\cdot\hat{\mathbf{d}}+\mathbf{f}_B\cdot\hat{\mathbf{d}}\Big)\mathbf{d}\cdot\mathbf{a}_0}\ . \qquad (43)$$

Of course, $\mathbf{f}_A+\mathbf{f}_B$ is the space part of the EM self-force given by (27), viz.,

$$F_{\rm self}=\frac{e_Ae_B}{d}\Big[(u\cdot\dot{v}_A)u-\dot{v}_A\Big]+O_{\rm SF}(d^0)\ ,$$

where $u=\gamma/d$, whence the space part $\mathbf{u}$ of u is what are now calling $\hat{\mathbf{d}}$ and $O_{\rm SF}(d^0)$ refers to the terms of order d^0 in the self-force expansion, which we have not calculated in this paper. We have

$$u\cdot\dot{v}_A=\frac{\gamma\cdot\dot{v}_A}{d}=-\frac{\mathbf{d}\cdot\mathbf{a}_0}{d}=-\hat{\mathbf{d}}\cdot\mathbf{a}_0\ .$$

Hence,

$$\begin{aligned}\mathbf{f}_A+\mathbf{f}_B&=-\frac{e_Ae_B}{d}\big[(\hat{\mathbf{d}}\cdot\mathbf{a}_0)\hat{\mathbf{d}}+\dot{\mathbf{v}}_A\big]+O_{\rm SF}(d^0)\\&=-\frac{e_Ae_B}{d}\big[(\hat{\mathbf{d}}\cdot\mathbf{a}_0)\hat{\mathbf{d}}+\mathbf{a}_0\big]+O_{\rm SF}(d^0)\ ,\end{aligned}$$

and

$$(\mathbf{f}_A+\mathbf{f}_B)\cdot\hat{\mathbf{d}}=-\frac{2e_Ae_B}{d}\hat{\mathbf{d}}\cdot\mathbf{a}_0+O_{\rm SF}(d^0)\ .$$

We thus have

$$\mathbf{f}_{\rm ext}^{\rm tot}\cdot\hat{\mathbf{d}}=2\left(m+\frac{e_Ae_B}{d}\right)\hat{\mathbf{d}}\cdot\mathbf{a}_0-\Big(\mathbf{f}_{\rm ext}^B\cdot\hat{\mathbf{d}}+\mathbf{f}_B\cdot\hat{\mathbf{d}}\Big)\mathbf{d}\cdot\mathbf{a}_0+O_{\rm SF}(d^0)\ .$$

We can drop the term in $\mathbf{f}_{\rm ext}^B$ that remains on the right-hand side, since we may assume that this force is $O(d^0)$, but we know that $\mathbf{f}_B$ is $O(d^{-2})$, so in principle we will get further terms of $O(d^{-1})$ and $O(d^0)$.

By (25), we have

$$F^\mu(A\text{ on }B)=e_Ae_B\left[\frac{u^\mu}{d^2}+\frac{-\dot{v}_A^\mu+(u\cdot\dot{v}_A)u^\mu}{2d}+O(d^0)\right]\ ,$$

whence

$$\mathbf{f}_B = e_A e_B \left[\frac{\hat{\mathbf{d}}}{d^2} - \frac{(\hat{\mathbf{d}} \cdot \mathbf{a}_0)\hat{\mathbf{d}} + \mathbf{a}_0}{2d} + O(d^0) \right] ,$$

and

$$\mathbf{f}_B \cdot \hat{\mathbf{d}} = e_A e_B \left[\frac{1}{d^2} - \frac{\hat{\mathbf{d}} \cdot \mathbf{a}_0}{d} + O(d^0) \right] .$$

So finally,

$$-\left(\mathbf{f}_{\mathrm{ext}}^B \cdot \hat{\mathbf{d}} + \mathbf{f}_B \cdot \hat{\mathbf{d}} \right) \mathbf{d} \cdot \mathbf{a}_0 = -e_A e_B (\hat{\mathbf{d}} \cdot \mathbf{a}_0) \left(\frac{1}{d} - \hat{\mathbf{d}} \cdot \mathbf{a}_0 \right) + O(d) .$$

So far then, we have

$$\boxed{\mathbf{f}_{\mathrm{ext}}^{\mathrm{tot}} \cdot \hat{\mathbf{d}} = \left(2m + \frac{e_A e_B}{d} \right) \hat{\mathbf{d}} \cdot \mathbf{a}_0 + e_A e_B (\hat{\mathbf{d}} \cdot \mathbf{a}_0)^2 + O_{\mathrm{SF}}(d^0) + O(d)} . \quad (44)$$

Let us try to interpret this equation. The total external force along the dumbbell axis in the ICIF is equal to the acceleration in that direction multiplied by a total mass equal to the observed mass $2m$ of the two charges augmented by $e_A e_B / d$, plus other terms that are independent of the dumbbell length or diminish at least linearly with it.

Of course, we must at least pay lip service to the terms $O(d^0)$, because we need some of them, viz., the left-hand side and the first term on the right-hand side. Note that the $O(d^0)$ terms in the EM self-force correspond to the part of that force that drives EM radiation when the system accelerates. Like the other $O(d^0)$ term, they go as the square of the acceleration. It would be interesting to calculate all these $O(d^0)$ terms, but that does not seem essential here. In his equation (19), Steane does not have terms of this order due to the dipole structure. It is well known that they are zero in the case of uniform acceleration.

The main point is the adjustment to the mass which would diverge in the point particle limit, viz., $e_A e_B / d$. This is what we would call the self-force-derived mass for the total self-force, and we note that it is equal to the energy-derived mass of the EM fields due to the dumbbell structure. We note also that the renormalised mass appears in a bona fide equation of motion for the dumbbell as a whole, namely (44), a crucial point for physical interpretation.

We have not yet evaluated the Poincaré stress (PS) self-force on the system, which we said would be $p(1)\mathcal{A} - p(0)\mathcal{A}$ along the instantaneous dumbbell axis $\hat{\mathbf{d}}$. The problem was that it was proportional to the unknown quantity $k\mathcal{A}$, which we eliminated in order to get the equation of motion (43). Adding (41) and (42) and rearranging somewhat, we obtain

$$\begin{aligned} \text{PS SF} + (\text{EM SF}) \cdot \hat{\mathbf{d}} &= m \mathbf{a}_0 \cdot \hat{\mathbf{d}} \left(1 + \frac{1}{1 + \mathbf{a}_0 \cdot \mathbf{d}} \right) - \left(2m + \frac{e_A e_B}{d} \right) \hat{\mathbf{d}} \cdot \mathbf{a}_0 \\ &\quad - e_A e_B (\hat{\mathbf{d}} \cdot \mathbf{a}_0)^2 - O_{\mathrm{SF}}(d^0) + O(d) \\ &= -\frac{e_A e_B}{d} \hat{\mathbf{d}} \cdot \mathbf{a}_0 + O(d^0) , \end{aligned}$$

where we have used (44), and the EM self-force here refers to its component in the $\hat{\mathbf{d}}$ direction. We conclude that the total self-force in the $\hat{\mathbf{d}}$ direction is

$$-\frac{e_A e_B}{d}\hat{\mathbf{d}}\cdot\mathbf{a}_0\;,$$

to leading order in d.

Now the EM self-force is

$$-\frac{e_A e_B}{d}\big[(\hat{\mathbf{d}}\cdot\mathbf{a}_0)\hat{\mathbf{d}}+\mathbf{a}_0\big]$$

to leading order in d, and this has component

$$-\frac{2e_A e_B}{d}\hat{\mathbf{d}}\cdot\mathbf{a}_0$$

in the $\hat{\mathbf{d}}$ direction. And since we know that the PS self-force is in the $\hat{\mathbf{d}}$ direction, we conclude that the PS self-force is indeed

$$\text{PS SF}=\frac{e_A e_B}{d}(\hat{\mathbf{d}}\cdot\mathbf{a}_0)\hat{\mathbf{d}}\;,$$

to leading order in d. Finally,

$$\text{total self-force}=-\frac{e_A e_B}{d}\mathbf{a}_0+O(d^0)\;. \tag{45}$$

Put another way, the PS self-force precisely removes the unwanted term in the EM self-force, so that to order d^{-1} the total self-force is indeed aligned with the acceleration $\mathbf{a}_0$ and the dynamical equations can keep the same form in the point particle limit by suitably renormalising the mass, provided that we keep the radiation terms of order d^0 mentioned above.

The above analysis also assumes that the sum of the EM force of A on B and the EM force of B on A should be taken in the way proposed in (24) on p. 14. To check this, we need to check that this sum appears in a suitable dynamical equation, i.e., a suitable version of Newton's second law, since at the end of the day this is the equation that must survive the point particle limit with a renormalised mass. But this is indeed the case, since we can interpret the renormalised mass from its role in (44), a crucial point here.

8 Another Way of Summing Forces Acting at Different Spacetime Events

The paper [9] by Ori and Rosenthal provides a resolution of the problem originally raised by Griffiths and Owen [4] and which has been rediscovered here. Indeed, bearing in mind that they consider a charge dumbbell with precisely the same motion as the one described in this paper, their fix works for the above calculation. In their notation, a factor of $\mathrm{d}\tau_B/\mathrm{d}\tau_A$ needs to be inserted with the first term on the right-hand side of (24) on p. 14. In this notation, the factor is $1-du\cdot\dot{v}_A(\tau_A)$. It perfectly cancels the unwanted part of the final expression (27) for the divergent term in the EM self-force. A full account of their method is given in Appendix A.

Of course, the important part of [9] is their justification introducing these factors. It does seem problematic that one added forces acting at different points in the charge distribution. Just after (9) on p. 9, we pointed out that we simply add the two forces together, even though they act at different points. This is not a problem in classical (non-relativistic) rigid body theory, partly because the body does not have different shapes in different frames, and partly because the forces can be considered to 'transmit' instantaneously to the centre of mass when assessing their translational effect on the body's motion.

But how can we formulate this aspect of things in the relativistic case? It does seem just plausible that things would be alright if one got the two forces simultaneously in the instantaneous rest frame of the dumbbell, and thanks to the rigidity assumption, there *is* an instantaneous rest frame for the whole dumbbell. But the real (physical) question is: what do we want the forces to do for us theoretically? What is their sum supposed to mean physically?

If A and B are the two charges, what we calculated above was the force of A on B plus the force of B on A, taking the second at the proper time τ_A of A and the first at the proper time $\tau_B(\tau_A)$ of B which corresponds to it, i.e., so that the two forces are taken simultaneously for the instantaneous rest frame of A. It does seem feasible that one should perhaps adjust the force on B to allow for some kind of delay in its being 'transmitted' to A, but that would amount to taking the force on B slightly earlier, which somehow spoils the idea that one ought more naturally to take the two forces simultaneously in the instantaneous rest frame of A. And it leaves open the question as to how the force on B might be transmitted to A, and how long that might take.

The idea in [9] is more subtle than this. In the present notation, they consider the energy–momentum change of A due to the force of B on A acting over a proper time lapse $\mathrm{d}\tau_A$ for A and they add the energy–momentum change of B due to the force of A on B acting over the corresponding proper time lapse $\mathrm{d}\tau_B$ for B, but keeping the calculation in the instantaneous rest frame:

$$F(A \text{ on } B)\mathrm{d}\tau_B + F(B \text{ on } A)\mathrm{d}\tau_A$$

So their idea is that, what is compatible with addition, or what it is useful to add up, in the instantaneous rest frame are infinitesimal changes in energy–momentum between the constant time hypersurfaces of consecutive instantaneous rest frames. It is then the rate of change of this with respect to the proper time at some preselected point, A in the present paper, that gives the required 'total' force:

$$F(A \text{ on } B)\frac{\mathrm{d}\tau_B}{\mathrm{d}\tau_A} + F(B \text{ on } A)\frac{\mathrm{d}\tau_A}{\mathrm{d}\tau_A} \tag{46}$$

This does indeed give a leading order term in the EM self-force that is aligned with the four-acceleration.

That's the fix, but we still have to understand physically why rates of change of this kind of summed energy–momentum should be a useful thing to consider. And that in turn depends on what we want this kind of total force to do for us. The answer is this: we are talking about (the relativistic version of) Newton's second law for the motion of the dumbbell as a whole, and we are looking for the extent to which these EM interactions within the dumbbell can contribute to its acceleration, or at least, to the acceleration of one point in it which we take to be representative of the acceleration of the dumbbell as a whole. In other words, we have to be quite clear that it is this total force that we expect to contribute to the acceleration of that point.

Is that obvious? Part of the key to this is equation (20) on p. 16 of [9], reproduced in the present paper as (74) on p. 53 (see Appendix A). Here we have a change in the energy–momentum of the dumbbell as a whole as it moves from the constant time hypersurface of the instantaneous rest frame at τ to the one at $\tau + \mathrm{d}\tau$. If we can justify this equation as the 'right thing', we are through, provided we can find a good physical interpretation for the quantity on the left-hand side. Steane discusses this possibility and concludes that the suggestion in [9] is sometimes a better, or rather a more natural definition, but not in every case [10].

The present view is that Ori and Rosenthal's account provides an alternative way of deducing the self-force effects in certain cases, but that those cases are not determined by what any observers are doing as suggested by Steane. Rather, their kind of analysis depends heavily on the equilibrium in the system being totally static, in a certain sense, something that is assumed for all the calculations discussed here anyway. As mentioned, their reasoning is presented in Appendix A. But first, it is interesting to examine Steane's assessment.

9 Comparing Different Definitions of Total Force

Steane begins with the general observation that the total four-momentum of a composite object should have the form [10]

$$p^{\mu}_{\mathrm{tot}}(\tau_{\mathrm{c}},\chi) = \sum_i p^{\mu}_i(\tau_{i,\chi}) \ ,$$

where χ is some spacelike hypersurface, i labels the components of the object, $\tau_{i,\chi}$ is the proper time on the worldline of the ith component when it crosses χ, and τ_{c} is the proper time on some arbitrarily chosen timelike worldline $x_{\mathrm{c}}(\tau_{\mathrm{c}})$ that serves as a reference, e.g., the worldline of the centroid of the object (Steane's notation here).

So a total four-momentum depends heavily on a choice of spacelike hypersurface. What seems natural to us humans is to choose a hyperplane of simultaneity, because we cling on to simultaneity despite what relativity theory (and especially general relativity) is intimating to us. But which hyperplane of simultaneity? As it turns out, it won't matter in situations where the object is isolated, in the sense that there are no 'external' forces operating on it [11, Sect. 5.2], because the sum will then always come out the same. But this is not at all the situation for the charge dipole, for two reasons. The first is that there *are* external forces ensuring its arbitrary acceleration, and the second is that it is being pushed around by its own EM field (perhaps not entirely an external force).

As Steane points out, when the object has rigid motion, we know that, for any chosen event on its world tube, there is an instantaneous rest frame for the whole thing in which the component occupying that event is at rest. Furthermore, the components always seem to one another to be the same distance apart in these instantaneous rest frames. This is of course the appeal of the rigid rest frames adapted to the motion of accelerating observers, e.g., the Rindler frame. So in the case of our charge dumbbell with its rigid motion, the obvious choice for χ in the above sum is one of these rest frames.

We thus see that there is already a non-obvious problem of definition when adding four-momenta. It is exacerbated when we wish to calculate four-forces, which are rates of change of four-momenta. The four-force on component i is

$$F_i^\mu := \frac{\mathrm{d}p_i^\mu}{\mathrm{d}\tau_i}\,,$$

so we might by analogy define the four-force on the composite object to be

$$\frac{\mathrm{d}p_{\mathrm{tot}}^\mu}{\mathrm{d}\tau_\mathrm{c}} := \lim_{\delta\tau_\mathrm{c}\to 0}\frac{p_{\mathrm{tot}}^\mu(\tau_\mathrm{c}+\delta\tau_\mathrm{c},\chi+\delta\chi)-p_{\mathrm{tot}}^\mu(\tau_\mathrm{c},\chi)}{\delta\tau_\mathrm{c}}\,,\tag{47}$$

taking the time τ_c to be the relevant time variable and assuming now that we have exactly one suitable spacelike hypersurface χ for each value of τ_c, and in such a way that $\delta\chi\to 0$ in some suitable sense as $\delta\tau_\mathrm{c}\to 0$. In other words, at least locally, we require a foliation on the spacetime in order to define the total four-force. But note that we still don't really know what this four-force is good for. What equation of motion does it feature in?

Steane considers two possible foliations on spacetime for his charge dipole:

- The hyperplanes of simultaneity of the inertial reference frame in which the dipole is currently at rest. This is what we have used in this paper and it is what Steane recommends when one considers a uniformly accelerating dipole in a flat spacetime without gravitational effects, not even non-tidal ones.
- The hyperplanes of simultaneity in the sequence of instantaneous rest frames of the charge dipole. It turns out that this corresponds to the recommendation by Ori and Rosenthal [9] and it is what Steane recommends when one considers a dipole that is somehow supported in a static homogeneous gravitational field.

We shall consider these two possible definitions in Sects. 9.1 and 9.2. The starting point in each case is (47), which can be written

$$\frac{\mathrm{d}p_{\mathrm{tot}}^\mu}{\mathrm{d}\tau_\mathrm{c}} := \lim_{\delta\tau_\mathrm{c}\to 0}\sum_i\frac{p_i^\mu\big(\tau_i(\tau_\mathrm{c})+\delta\tau_i\big)-p_i^\mu\big(\tau_i(\tau_\mathrm{c})\big)}{\delta\tau_\mathrm{c}}\,,$$

where $\delta\tau_i$ is the change in τ_i in going from the hyperplane of simultaneity through the event $x_\mathrm{c}(\tau_\mathrm{c})$ to the hyperplane of simultaneity through the event $x_\mathrm{c}(\tau_\mathrm{c}+\delta\tau_\mathrm{c})$, and the hyperplanes of simultaneity are those of the chosen foliation. Hence, we can write

$$\frac{\mathrm{d}p_{\mathrm{tot}}^\mu}{\mathrm{d}\tau_\mathrm{c}} = \lim_{\delta\tau_\mathrm{c}\to 0}\sum_i\frac{\mathrm{d}p_i^\mu}{\mathrm{d}\tau_i}\frac{\delta\tau_i}{\delta\tau_\mathrm{c}}\,.\tag{48}$$

We shall also make some remarks about the problem of defining physical quantities in so-called accelerating frames of reference in Sect. 10, when we discuss Steane's analysis of the self-force on the dipole in a static homogeneous gravitational field (SHGF).

9.1 Instantaneously Comoving Inertial Frame

For the very special kind of situation of rigid motion that we are considering, we know that, for all values τ_{c0} of τ_c, there is an instantaneously comoving inertial frame ICIF(τ_{c0}) for all components i, i.e., in which all components i are instantaneously at rest. Then each χ in the foliation is specified by a coordinate time t in this instantaneously comoving inertial frame, and $\chi + \delta\chi$ will be specified by some other coordinate time $t + \delta t$ in that same frame, in the sense that χ and $\chi+\delta\chi$ are the hyperplanes of simultaneity corresponding to those coordinate times.

So for each value of τ_c near the value τ_{c0} where this frame is the ICIF, there is a corresponding value of t in that ICIF and there is a corresponding value of τ_i, and $\delta\tau_i$ in (48) is the change in τ_i in going from the hyperplane of simultaneity of ICIF(τ_{c0}) specified by τ_c, which corresponds to some specific value of t, to the one specified by $\tau_c + \delta\tau_c$, which corresponds to $t + \delta t$. Furthermore, for $\tau_c = \tau_{c0}$, we will have $\delta\tau_i = \delta t = \delta\tau_c$ for all components i because our fixed inertial frame happens to be this ICIF. We then obtain Steane's equation (30), viz.,

$$\frac{\mathrm{d}p^\mu_{\mathrm{tot}}}{\mathrm{d}\tau_c} = \sum_i \frac{\mathrm{d}p^\mu_i}{\mathrm{d}\tau_i} \,, \tag{49}$$

which says that the total four-force defined in this way is equal to the sum of the four-forces on all the components. Note the proviso though: the fixed inertial frame has to be the ICIF. Later, when the system gets moving relative to this frame, (49) is not generally true.

This shows just how careful one has to be when adding up forces on different parts of an extended body, especially if it does not have rigid motion, in which case there will be no inertial frame in which the simple sum of four-forces will be equal to the total self-force as defined by (47). Finally, note that, for rigid motion and when the fixed inertial frame is taken to be an ICIF, we also have

$$\frac{\mathrm{d}p^\mu_{\mathrm{tot}}}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}p^\mu_i}{\mathrm{d}t} \,, \tag{50}$$

and in particular,

$$\frac{\mathrm{d}\mathbf{p}^\mu_{\mathrm{tot}}}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}\mathbf{p}^\mu_i}{\mathrm{d}t} \,, \tag{51}$$

which is Steane's equation (29).

We can raise plenty of questions here. First note that in the pre-relativistic dynamics of rigid bodies we could build up a whole theory of motion in which the total force defined in the obvious way had an obvious role. But what is the role of the object defined in (47)? Why is it good for our total self-force to equal that? In what way does it determine the motion of the system as a whole? These are questions that have to be answered at least approximately in each particular case, as we have done for example at the very end of Sect. 7 (see p. 26).

9.2 Rigid Accelerating Frame

Steane is considering Rindler frames in [10]. These are examples of rigid accelerating frames of the kind described in detail in [8]. Such a frame can be adapted to the motion of an observer with arbitrary acceleration, in the sense that the observer remains forever at the space origin of the frame and the time coordinate is her proper time. Furthermore, worldlines tracked by events with constant space coordinates remain at the same proper distance from similar points nearby as measured in their own instantaneous rest frames. (This is what is rigid about such frames.) In the present paper, the charge dumbbell occupies events with constant space coordinates in such a frame. Furthermore, the frame in question has as little rotation as possible relative to some fixed inertial frame. The Rindler frame is a special case where the observer has uniform acceleration in a straight line.

The foliation used to define the total four-force is here taken to be the sequence of instantaneous rest frames of the extended object, which is just the sequence of hyperplanes of simultaneity of the rigid accelerating frame, labelled by the proper time of the observer, if there is one, or the proper time of some selected material point in the object. This time (48) reads

$$\frac{\mathrm{d}p^{\mu}_{\mathrm{tot}}}{\mathrm{d}\tau_{\mathrm{c}}} = \lim_{\delta\tau_{\mathrm{c}}\to 0} \sum_i \frac{\mathrm{d}p^{\mu}_i}{\mathrm{d}\tau_i}\frac{\delta\tau_i}{\delta\tau_{\mathrm{c}}} = \sum_i \frac{\mathrm{d}p^{\mu}_i}{\mathrm{d}\tau_i}\frac{\mathrm{d}\tau_i}{\mathrm{d}\tau_{\mathrm{c}}}\ , \tag{52}$$

where $\tau_i(\tau_{\mathrm{c}})$ is effectively the function $\tau_s(\tau_A)$ mentioned just prior to (33), with suitable adjustment of notation. By comparison with (46) on p. 27, we see immediately that Ori and Rosenthal's theory in [9] corresponds precisely to this.

So as mentioned in connection with (46), what is compatible with addition, or what it is useful to add up, in the instantaneous rest frame are infinitesimal changes in energy–momentum between spacelike hypersurfaces, but *not between just any* spacelike hypersurfaces, and in fact, not between hyperplanes of simultaneity of a single instantaneous rest frame, but between constant time hypersurfaces of consecutive instantaneous rest frames for the rigid system. It is then the rate of change of this with respect to the proper time at some preselected point, A in the present paper, that gives the required 'total' force (46) on p. 27.

9.3 Summary

So just to recapitulate, when a composite object is not isolated, and this is the case for the charge dipole here because it is being pushed around by the external force which imposes the acceleration *and* by its own EM field, we know that p_{tot} must depend on a choice of spacelike hypersurface, and any time derivative of it interpreted as a force must depend on a local foliation. Each foliation gives a possible definition, and the whole problem here becomes one of understanding what equation of motion that choice of foliation delivers.

The equation of motion is important, because we use it to find out where the dipole goes, and that must be the same whichever of the two definitions we use. So this would provide a point of comparison. It is thus crucial to have some explicit discussion of that, for without it, how could we really assess the physical significance of these definitions?

10 Self-Force on the Dipole in an SHGF

In the abstract of [10], Steane states that an observer fixed in a gravitational field described everywhere by the Rindler metric will find any charged object supported in the gravitational field to possess an EM self-force equal to that observed by an inertial observer relative to which the body undergoes rigid hyperbolic motion. He concludes from this that some form of equivalence principle (EP) is satisfied by these systems, but we should note that we do not even know how to do electromagnetism in the context of general relativity until we apply the strong equivalence principle (SEP), because this tells us how we can ship our theories of non-gravitational physics into the gravitational context provided by GR [5].

By 'supported in the gravitational field', he means sitting at fixed coordinates in the Rindler frame. But sitting at fixed space coordinates in the Rindler frame means moving with constant acceleration, i.e., hyperbolic motion, in a freely falling Minkowski frame. If the object is also moving rigidly, this is technically exactly the same as the same object moving rigidly with constant acceleration in an inertial frame. Furthermore, if the Rindler situation is taken to be a GR description of a static homogeneous gravitational field, and provided we invoke SEP to get electromagnetic theory in that context, the EM fields will be precisely the same in the two different physical situations, and so will all EM forces the object exerts on itself or anything else.

But Steane's claim is that a Rindler observer, i.e., one sitting at fixed Rindler coordinates, which he interprets to mean 'fixed' in the SHGF, will 'find' any charged object that is also fixed in the gravitational field in this sense to have an EM self-force equal to that 'observed' by an inertial observer in this same gravitational context. This is a reference to an accidental equality that Steane has identified in his equation (36). If F is the covariant Faraday tensor and u the four-velocity of a test particle as expressed in Rindler coordinates, and if $\overline{F}$ and $\overline{u}$ are the same quantities as expressed in an inertial frame that is instantaneously comoving with the test charge, then it is easy to check that

$$\overline{F}_{\mu\nu}\overline{u}^{\nu} = F_{\mu\nu}u^{\nu} \ . \tag{53}$$

This is a purely numerical equivalence that just happens to work because of the form of the transformation from inertial to Rindler coordinates. But since Fu is the Lorentz force on the test charge in the EM field described by F, we deduce with Steane that, for any given EM field in flat spacetime, the components of the EM four-force on a particle sitting at fixed Rindler coordinates, expressed in the coordinate system of that frame, are the same as those of the EM four-force on that same particle, expressed in the coordinate system of a Minkowski frame relative to which the Rindler frame is momentarily at rest.

So this is not the equivalence of two physical situations, but the accidental equality of two quantities calculated in different coordinate systems, and it would of course work for either of the mathematically equivalent physical situations discussed above, viz., a flat spacetime with no gravitational field, not even a non-tidal one, and an SHGF. It is accidental because one would not normally have an equality like (53) between two covariant vectors expressed in different coordinate systems.

So what exactly is the content of Steane's claim? What he is saying is that, in addition to the physical equivalence of two different situations that is ensured by SEP, and which he assumes implicitly, the EM self-force is actually *numerically equal*, despite being a four-vector, in the two coordinate formulations. From this he can conclude that all his earlier statements about EM forces on an accelerating dipole in a flat spacetime without gravitational effects, not even non-tidal ones, also apply to EM forces on a dipole sitting at fixed Rindler coordinates in an SHGF described by the Rindler form of the Minkowski metric, that is, even when those forces are expressed in Rindler coordinates.

In particular, for the self-force of any arbitrary charge distribution undergoing *rigid* acceleration, the EM self-force is the same in the Rindler coordinates as in the Minkowski coordinates of the instantaneous inertial rest frame. Furthermore, this is true whether we define the self-force according to either (49) or (52). We do need to check this, and in particular to see why the motion must be rigid for this to work.

Consider first the sum of EM forces in (49). The forces in the sum apply at different spacetime events, so different values of Steane's Rindler coordinates (θ, h). Having different values of θ would be a problem because the Lorentz boost part of the transformation to Rindler coordinates which Steane identifies is θ dependent, so we have to arrange for each term in the sum of (49) to correspond to the same θ. This works because we sum for events in a common ICIF of all the particles, something we always have for rigid motion. But each common ICIF has the form $\theta =$ constant (hypersurface in spacetime). In Steane's coordinates, an example is $T = 0$, when $\theta = 0$.

Regarding the second possible definition of the self-force, viz., (52), which corresponds to Ori and Rosenthal's method in [9], its components will also agree numerically in the Rindler frame and in the ICIF of the rigidly moving charge distribution because the extra factors we now have, viz., $\mathrm{d}\tau_i/\mathrm{d}\tau_\mathrm{c}$, do not depend on any coordinates.

Steane now considers the two cases (49) and (52) and, without physical justification, declares the second to be the more natural definition of the total force on an extended object when it is 'supported' in a gravitational field.

We shall consider that claim in detail and try to raise problems with it. But first, what about the definition (49).

10.1 Defining Self-Force with a Minkowski Foliation

If we adopt the definition (49) based on a Minkowski foliation of constant time hyperplanes in in the instantaneously comoving inertial frame, but then evaluate it in Rindler coordinates, we find exactly what we would have found if we had evaluated it in inertial coordinates in the ICIF, so we conclude for the Rindler coordinate version that the EM self-force is greater when the dipole is oriented along the gravitational field than when it is oriented transverse to it. The numbers are all exactly the same as we had before!

As Steane puts it, in view of the pressure forces in the rod, we must have the above discrepancy between the EM self-forces for longitudinal and transverse acceleration if his EP is to be satisfied. The extra self-force in the longitudinal case is supplying the necessary pressure forces in that case. Put another way, he declares that the weight of a dipole is independent of its orientation. But what EP requires that?

Let us suppose that Steane's EP is what he calls the weak EP in [11], i.e., equality of PGM and inertial mass. Note that we must reinstate the PGM in the general relativistic context. One way to do this is described in [7, Chap. 4]: we can take it to be the constant of proportionality between four-acceleration and the four-force required to hold the object at fixed Rindler space coordinates. But this EP would not be affected by an orientation dependence when we add field masses to 'mechanical' masses if we apply this reinstatement of the PGM. The point is that, thanks to SEP, which is always necessary if we want to say anything about non-gravitational forces in the GR context, *each* of the EM self-force and the Poincaré stress self-force contributes equally to this PGM and also to the inertial mass. Nothing can go wrong here because we have to use SEP, and SEP delivers this by decree. There is nothing to prove.

So a lot depends on how Steane proposes to reinstate the PGM in GR. Elsewhere, he states that the PGM of the complete matter plus field system is determined by energy, whereas the inertial mass is determined by momentum. The idea here is that, in GR, it is energy that is attracted gravitationally, and the energy-derived mass of the fields is equal to the self-force-derived mass only for transverse motion. But is the energy of the fields the same for arbitrary longitudinal and transverse motion? That seems unlikely. The energy-derived mass is calculated for the stationary dipole. In any case, if this is how one should reinstate the PGM, the problem is not that the self-force-derived mass should differ for different orientations, but rather that the self-force-derived mass should differ from the energy-derived mass.

10.2 Defining Self-Force with a Rindler Foliation

Having deduced everything we could hope for about the dipole sitting at fixed Rindler coordinates in the SHGF, even when physical quantities are expressed relative to Rindler coordinates rather than inertial ones, we now deduce everything again in a different way, by switching from the definition (49) of the self-force, which uses a Minkowski foliation of spacetime, to the Ori–Rosenthal definition (52), which uses a Rindler foliation. This really is a different way quantitatively, in the details, but the qualitative conclusions can be made to work out the same as we shall now see.

Steane first needs to define what should be the electric field in an arbitrary frame, then apply this definition to the electric field of a uniformly accelerating charged particle fixed at an arbitrary space point of a suitable Rindler frame, expressing his electric field as a function of the corresponding Rindler coordinates. He goes about this in an indirect way that appeals to his trick (53).

Obviously, to apply the trick, he suggests defining the electric field for a Faraday tensor F in an arbitrary frame to be the spatial part of the four-force per unit charge on a charged particle that is 'not moving' relative to that frame, i.e., sitting at fixed space coordinates. Now the Lorentz four-force on such a particle is

$$\mathcal{E}^i := F^i{}_\lambda u^\lambda \;, \qquad u^\lambda := \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} \;,$$

where τ is the proper time of the particle, $x^\lambda(\tau)$ its worldline, and $u(\tau)$ its four-velocity. Note that Steane describes u as the four-velocity of the 'local observer fixed in the frame'. Of course, she must be fixed with the charge. If the charge is not where the observer is, we have a problem! We need observers fixed everywhere in the frame, so this is no longer the frame of *an* observer, but the frame of a continuum of observers.

This means that

$$u(x) = \frac{\mathrm{d}t}{\mathrm{d}\tau}(1,0,0,0) \;,$$

where t is the time coordinate x^0. Hence,

$$\boldsymbol{\mathcal{E}} = (F^1{}_0, F^2{}_0, F^3{}_0)\frac{\mathrm{d}t}{\mathrm{d}\tau} \;,$$

or

$$\mathcal{E}^i = (-g_{00})^{-1/2} F^i{}_0 = (1+gx)^{-1} \;,$$

when the metric interval has the particular Rindler form

$$\mathrm{d}s^2 = -(1+gx)^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \;. \qquad (54)$$

Note that, for this particular metric,

$$F_{i0} = g_{i\mu}F^\mu{}_0 = g_{ii}F^i{}_0 \text{ (no sum)} = F^i{}_0 \;,$$

using the antisymmetry of F, so we also have

$$\mathcal{E}^i = (-g_{00})^{-1/2}F_{i0} = -(1+gx)^{-1}(\text{first row of } F_{\text{cov}}) \;.$$

Of course, one could just take the 'electric field' in this frame to be the first row of F_{cov}. We can define things how we like, but at some point we have to say what it means physically. More about this later.

To get his electric field in the Rindler frame, Steane begins by writing down the standard electric field $\overline{\boldsymbol{E}}$ due to a uniformly accelerating charge, as expressed relative to Minkowski coordinates in an inertial frame that is instantaneously comoving with the charge. He effectively notes that this corresponds to a special case of his generalised definition of an electric field, then applies his trick (53) to deduce that

$$\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{E}} \; .$$

So we have $\boldsymbol{\mathcal{E}}$ as a function of inertial coordinates and convert to a function of the above Rindler coordinates (t, x, y, z). The uniform acceleration of the source charge that appeared in the formulas for $\overline{\boldsymbol{E}}$ is obtained by insisting that it match that of the 'local observer' in the frame with the above metric interval (54), whence it gets expressed in terms of the parameter g.

So we now have our generalisation of the notion of electric field 'for' the Rindler frame, or 'for' the Rindler observers, however one would like to put that. It is just a definition. So far, we don't know what good it is to us. Note that we could have just converted the Minkowski version of F to a Rindler version by the usual coordinate transformations and then read off this definition of a Rindler electric field.

Steane notes that, if we use this $\boldsymbol{\mathcal{E}}$ to calculate the self-force for a rigidly moving charge distribution, the results will agree with those found in the Minkowski frame. This is clear, given the accidental equality (53) and having just defined $\boldsymbol{\mathcal{E}}$ so that this equality will apply. Of course, he refers here to the EM self-force calculation using the definition (49) for the sum of forces applying at different locations. But as mentioned, he does not consider that to be the *most natural* way to form such a sum of forces.

In the present context where we imagine ourselves to be Rindler observers sitting at fixed Rindler space coordinates in a gravitational field, i.e., 'supported' in an SHGF rather as we are on the surface of the Earth, Steane considers it better to sum the forces as laid down in (52) on p. 31, i.e., the Ori–Rosenthal way. That is, the hyperplane $\chi + \mathrm{d}\chi$ is taken to be the next hyperplane of simultaneity as defined by the accelerating frame. Of course, that sounds natural, but all hyperplanes of simultaneity are factitious. Accelerating observers are supposed to borrow them from ICIFs.

We still have to choose a reference worldline, the one with proper time τ_{c} mentioned earlier, which we originally suggested to be the worldline of the system centroid. But now, according to Steane, if we wish to 'study dynamics more generally', we must select a reference worldline that is independent of the objects under consideration. He thus opts for the worldline of a point fixed in the frame, by which he means a worldline with fixed space coordinates, and in particular, the point where g_{00} takes the value -1. This is once again the *most natural* one for Steane, because $\mathrm{d}\tau_{\mathrm{c}} = \mathrm{d}t$. In fact, t is the proper time for this point. We think of the frame as being adapted to an observer following this worldline, hence set up by this observer.

But our object is moving rigidly and all its particles have the requisite uniform acceleration for this, so each point in it sits at a fixed space point of the Rindler frame. This makes the above motivation seem rather weak. We note that the reference worldline is characterised by $x = 0$, and an observer sitting there would have uniform acceleration g. This does of course fit with the interpretation of a frame somehow supported in an SHGF with 'acceleration due to gravity' equal to g, but all worldlines with other fixed values of x have *different* accelerations, so the picture of a 'supported' frame in such an SHGF is already tarnished. Note also the scare quotes on 'acceleration due to gravity', because we are using GR here and there is no such thing in GR (see the discussion in [5]).

The upshot of the above choice is that we get the values of $\mathrm{d}\tau_i/\mathrm{d}\tau_{\mathrm{c}}$ that are needed in the calculation of (52), even if interpretation remains a major issue. If particle i is the one sitting fixed at Rindler coordinate value x, then

$$\frac{\mathrm{d}\tau_x}{\mathrm{d}\tau_{\mathrm{c}}} = \sqrt{-g_{00}(x)} = 1 + gx \; . \tag{55}$$

Then the force per unit charge that must be summed in order to calculate our self-force according to (52) is given by something that Steane denotes suggestively by E^i, viz.,

$$E^i := \sqrt{-g_{00}(x)} F^i{}_\lambda u^\lambda = F^i{}_0 \; ,$$

since

$$\sqrt{-g_{00}(x)} u(x) = (1,0,0,0) \; .$$

We note that this is

$$E^i = (1 + gx)\mathcal{E}^i \; .$$

Of course, this is another possible definition of the electric field in this case (see Sect. 11).

Steane now obtains formulas for $\boldsymbol{E}$ using the ones for $\boldsymbol{\mathcal{E}}$. If the positions of charges A and B are

$$\mathbf{r}_A := (x_A, y_A, z_A) \; , \qquad \mathbf{r}_B := (x_B, y_B, z_B)$$

in Rindler space coordinates (x, y, z), he then notes that the EM self-force according to (52) is

$$\mathbf{f}_{\mathrm{self}} = -q_A q_B \big[\mathbf{E}(\mathbf{r}_A, \mathbf{r}_B) + \mathbf{E}(\mathbf{r}_B, \mathbf{r}_A)\big] \; ,$$

where $\mathbf{E}(\mathbf{r}_A, \mathbf{r}_B)$ is the $\mathbf{E}$ field at $\mathbf{r}_B$ due to the charge at $\mathbf{r}_A$, and $\mathbf{E}(\mathbf{r}_B, \mathbf{r}_A)$ is the $\mathbf{E}$ field at $\mathbf{r}_A$ due to the charge at $\mathbf{r}_B$. We can write this down exactly using the formulas at hand. Expanding in powers of

$$d := |\mathbf{r}_B - \mathbf{r}_A| := \big[(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2\big]^{1/2}$$

and keeping only terms up to $O(g^4)$ in g, we find

$$\mathbf{f}_y^{\mathrm{self}} = O(d) \; , \qquad \mathbf{f}_z^{\mathrm{self}} = O(d) \; ,$$

and

$$\mathbf{f}_x^{\mathrm{self}} = \frac{q_A q_B}{d} \left\{ g + \frac{3d^2 - \Delta x^2}{8} \big[-g^3 + (x_A + x_B) g^4 \big] \right\} \; , \tag{56}$$

where $\Delta x := x_B - x_A = O(d)$.

This means that there are no $O(d^{-1})$ terms in $\mathbf{f}_y^{\text{self}}$ and $\mathbf{f}_z^{\text{self}}$. These are the terms that would diverge in the point particle limit, and because Steane's acceleration is in the x direction, we would not be able to absorb them in a mass renormalisation move. Of course, we knew these would have to be zero because we have already seen that Ori and Rosenthal's fix ensures that the leading order term in the self-force will be aligned with the acceleration, for *any* acceleration of the charge dipole, not just the uniform acceleration case considered by Steane. But the $O(d^0)$ terms in $\mathbf{f}_y^{\text{self}}$ and $\mathbf{f}_z^{\text{self}}$ are also zero. Steane suggests that these components are actually zero to all orders in d, which sounds surprising, especially when one looks at the calculations involved.

Anyway, this is the orientation independence of the EM self-force in this picture, at least for uniform accelerations, i.e., the divergent term in the EM self-force (for the point particle limit) is always aligned with the acceleration in this 'picture'. It implies that classical renormalisation is possible for point particles, here when the length d of the dipole tends to zero, provided that the redefinitions involved in formulating this picture 'for' the uniformly accelerating observer lend themselves to a physically understandable equation of motion, something that is not discussed in Steane's paper but gets full coverage by Ori and Rosenthal (see the account in Sect. A.4). The divergent term in the x component of the self-force is $q_A q_B g/d$, so if we are not too critical, it looks as though the self-force-derived mass will be $q_A q_B/d$, just as we would hope (but see below).

Finally, the $O(d)$ term in the x component of the self-force is zero, so there is no $O(d)$ term in any component of the self-force. This might be interpreted as saying that there is no radiation due to interaction between the two charges in the dipole for this case of uniform acceleration, at least 'for' the comoving observer, a much debated issue [5]. But note that it is not immediately clear how to interpret these results, which appear to involve somewhat arbitrary redefinitions of physical quantities 'for' the co-accelerating observer.

So in some sense, classical electromagnetism can be renormalised for Rindler observers of comoving charge dipoles, due to the alignment of the EM self-force in this picture and the uniform acceleration of the charge dipole. But what about Poincaré stress self-forces? Steane models the Poincaré stresses by pressure. First we show that

$$p \propto \frac{1}{1+gx} \tag{57}$$

when calculated in the ICIF. We use the Rindler space coordinate x, which is just borrowed directly from the ICIF [8]. We know that a particle fixed at x has proper acceleration $g/(1+gx/c^2)$, so the relativistic version of the Navier–Stokes equation (31) reads

$$\frac{p}{c^2}\frac{g}{1+gx/c^2} = -\frac{\mathrm{d}p(x)}{\mathrm{d}x}\ ,$$

when $\rho \sim 0$, even when the rod is not aligned with the x axis. This has the solution (57).

The important thing now is to understand that the pressure 'for' the uniformly accelerating observer will go as $p(1 + gx/c^2)$, that is, it will be independent of x. The point is that, to get a force in the Rindler frame as defined by Ori and Rosenthal in their scheme (52), we have to multiply the force in the Minkowski frame by $\mathrm{d}\tau_x/\mathrm{d}\tau_\mathrm{c}$ as given by (55). Of course, if the pressure is constant along the rod, the associated self-force due to Poincaré stresses will be zero, i.e., internal forces cancel, for the Ori–Rosenthal definition of forces 'for' accelerating observers.

We conclude that, in the picture we have devised 'for' uniformly accelerating observers, the total self-force, equal to the EM self-force plus PS self-force, is just the EM self-force alone, viz., $q_A q_B g/d + O(d^0)$, always aligned with the acceleration, whatever the orientation of the dipole axis. So qualitatively, we obtain the same result in this 'Rindler picture' as we did in the 'Minkowski picture', although it is not obvious why that should be. Further, we know that this works for arbitrary accelerations of the charge dipole from the first part of this paper. Suppose it works for arbitrary charge distributions and arbitrary accelerations. What would that mean? Could we conclude that the Ori–Rosenthal picture is the right one 'for' the co-accelerating observer?

It is striking that the acceleration g in the formula (56) for f_x^{self} is not the acceleration of the dipole! It is the acceleration of an observer sitting fixed at Rindler coordinate $x = 0$. In a sense it might be taken as the acceleration of the frame, because the frame is adapted to this worldline, i.e., the time coordinate is the proper time of an observer following this worldline. However, all other worldlines at different fixed values of x have different accelerations $g/(1 + gx)$.

So why should $q_A q_B g/d$ be relevant here? After all, we could have placed our observer anywhere. Don't we want f_x^{self} as a function of the acceleration of the rod? This makes the formula (56) look rather arbitrary. Steane says that, in order to compare it with the formula for the self-force he calculated in the Minkowski picture, which has leading order term $q_A q_B a/d$, with a the acceleration of the rod, one must divide (56) by

$$\sqrt{-g_{00}(x_0)} = 1 + gx_0 \ ,$$

where $x_0 := (x_A + x_B)/2$ is a kind of estimate of the point we should take in the rod to obtain its acceleration, i.e., we estimate that the rod has acceleration $g/(1 + gx_0)$, even though we know that each point in it has a slightly different acceleration.

Of course, the rod may have a very different acceleration to g so this looks promising in a way. The point is that (56) is a self-force in the Ori–Rosenthal picture specified by (52) on p. 31, so we expect something like

$$\begin{aligned} f_{\mathrm{self}}(\text{O/R picture}) &= f_{\mathrm{self}}(\text{fixed inertial frame}) \times (1 + gx_0) \\ &= \frac{q_A q_B}{d} \times \text{acceleration of dipole} \times (1 + gx_0) \ , \end{aligned}$$

then divide by $1 + gx_0$ to convert to a self-force as it would normally be calculated in an instantaneously comoving inertial frame, according to (49) on p. 30. This does work, because the acceleration of the dipole is $g/(1+gx_0)$.

So we have some kind of numerical agreement between the two pictures, at least for the leading order term in the self-force expansion. However, it looks very unlikely that this will prevail to all orders, simply because there is no obvious reason why it should, in the details. Could one just put that down to the approximations involved? Or is it not important for these different observers to make numerically different predictions in their 'pictures'?

Just after noting that the leading order term in (56) agrees exactly (sic) with the result for an accelerating dipole observed by an inertial observer, Steane concludes that the system satisfies EP. Hopefully, this is not intended to be some deep claim about what the Rindler observer sees, perhaps along the lines that her qualitative picture has similarities with the one that might be obtained by an inertial observer. In pre-relativistic Newtonian gravity, there was an 'equivalence principle' a bit like this, viz., an observer cannot tell by looking at some objects all moving away with the same acceleration whether she is accelerating or whether she is fixed in absolute space and the objects are falling freely in a gravitational field (although EM effects mess that up [5]). Of course, this is because the PGM of Newtonian gravity is exactly equal to the inertial mass.

Presumably, Steane intends the latter version of EP. But that has nothing to do with whether accelerating observers might think they were not accelerating [5]. It is thus odd that he concludes to non-violation of his EP just after observing that lowest order terms 'agree' in the sense described above, when we use a fixed inertial frame or the Ori–Rosenthal fix. Why would that reflect on the equality of the PGM and the inertial mass?

It could be that Steane has in mind something like the argument for a *mechanism* for this equality exposed in [7, Chap. 4]. However, he does not formulate that here. It seems significant that he accepts to leave 'the acceleration of the frame' g in his formula for f_x^{self}, since this is the 'acceleration due to gravity'. What we get then is a 'picture' for the observer 'supported' in the gravitational field. In this picture, the 'acceleration due to gravity' causes a self-force on the charge distribution which contributes $q_A q_B/d$ to its PGM, exactly the EM self-force contribution to the inertial mass.

Still it seems almost too good to be true that the leading order term in the EM self-force is $q_A q_B g/d$ in this picture, where the acceleration g of the observer could be anything, i.e., it seems to be quite independent of the acceleration of the dipole. The parameter g is entirely determined by what the observer is doing. The dipole only has to be moving rigidly with respect to this observer, but it could be anywhere provided it satisfies this constraint, and it could thus have any acceleration whatever. The real problem then seems to be that Steane's f_{self} is a function of an arbitrary parameter g. Would he then say that g has to appear here because this is a picture for an observer with this acceleration?

Perhaps we ought to ask how we know what this Rindler observer will find. There is of course an assumption here that an observer with this motion in the SHGF will use Rindler coordinates to express physical quantities. It should be borne in mind, however, that there are no canonical coordinate systems for accelerating observers, even though there may be some theoretically comforting ones or convenient ones [5].

We ought also to ask why we should be interested here in what the Rindler observer will find. Why bother with an accelerating observer? Is it not enough to analyse the situation in the freely falling frame? But then we ought to be satisfied with the calculations already done, since the theoretical scenario is identical, provided we use SEP to get electromagnetic theory in the context that is considered to be gravitational. Why do we need a 'picture' for the observer 'supported' in the gravitational field?

The obvious answer is that observers like ourselves are in fact supported in gravitational fields. The aim when people introduce Rindler observers, or other such Killing observers following the flow curves of a Killing vector field in a gravitational field with a symmetry of some kind, is really to make a 'picture'. This is supposed to be what the observer will see, or find, or think about physical quantities. And yet there is never any serious discussion of the operational aspects of what such observers would do if they were out there, and without that, we may as well do calculations without any reference to observers.

The above comments are made because the whole of Steane's discussion here seems to suggest that Rindler observers will make a certain kind of 'picture', and that it is useful to know what it is, when in fact there is no real attempt to justify those claims physically, and in any case we already have one 'picture' in the freely falling frame which we know by decree (SEP) to be exactly equivalent. In a gravitational context, we must apply SEP, otherwise we don't even know how to formulate electromagnetism. This sets up the non-gravitational theory, the minimal extension of Maxwell's equations (MEME) to the GR context, in the freely falling frame.

In the present case this is straightforward because there is a global inertial frame, and it is precisely what Steane does (effectively) to get his electric field $\boldsymbol{\mathcal{E}}$ for the uniformly accelerating charge in Rindler coordinates: he assumes that we know the field in Minkowski coordinates and effectively carries out a straightforward coordinate transformation. But there is no consideration of the accelerating (supported) frame we might actually set up, say using standard measurement techniques, i.e., there are no operational definitions.

Why would a uniformly accelerating observer set up a Rindler frame? Her rulers would have to satisfy the ruler hypothesis [6]. She would have to borrow the hyperplanes of simultaneity from instantaneously comoving inertial observers. She would attribute her own proper time to those hyperplanes. That may seem reasonable enough, but there is nothing canonical about it. There is no deep physical meaning in those coordinates. They are just coordinates. Would it not be better merely to predict how each accelerating system should behave according to MEME and drop the pictures?

11 Defining the Electric Field in an SHGF

There is some discussion in Steane's paper of the advantages and disadvantages of various definitions of the electric field. The position adopted in the present paper is that the only relevant considerations when it comes to definitions are technical ones regarding ease of calculation, i.e., no physical meaning can really be attached to 'pictures' that may or may not be adopted by observers with different kinds of motion, unless perhaps there is some discussion of the operational aspects (and there is none in Steane's paper, or any other paper making such claims, to the author's knowledge).

As above, one possible definition of the electric field and magnetic field are the space parts of the four-force per unit charge, viz.,

$$\mathcal{E} := u^\lambda F^i{}_\lambda \ , \qquad \mathcal{B}^i := \frac{1}{2}\varepsilon^i{}_{\lambda\mu\nu} u^\lambda F^{\mu\nu} \ ,$$

in Steane's notation. Then for a particle that is not moving relative to the space coordinates, we have

$$u = \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}(1,0,0,0) = (-g_{00})^{-1/2}(1,0,0,0) \ ,$$

so

$$\mathcal{E}^i = (-g_{00})^{-1/2} F^i{}_0 \ .$$

For a diagonal metric tensor with ones on the space part of the diagonal, this is

$$\mathcal{E}^i = (-g_{00})^{-1/2} F_{i0} = (-g_{00})^{-1/2} \times \begin{pmatrix} \text{space part of first} \\ \text{column of } F_{\mathrm{cov}} \end{pmatrix} \ .$$

But why should we choose this rather than, say, F_{i0}? Could there really be a good physical justification?

On the face of things, it sounds like a reasonable idea to take the electric field as the force per unit charge, at least for the case where the particle is 'not moving', because that is what we do in an inertial frame. (Not moving just ensures that there is no magnetic effect, in the case of the inertial frame.) But what does it mean physically to be 'not moving' in some non-inertial frame? Certainly, nothing in an arbitrary frame, and in the Rindler frame it means moving with a quite different acceleration to the 'frame', i.e., to the person who chooses this frame according to the usual rigid construction.

There is at best very little discussion about this, even in the case where the Rindler frame is supposed to be the frame of an observer suspended in a gravitational field. It is often just stated that the observer is not moving. But the gravitational field in this case is non-tidal (zero curvature), and there is no mention of the matter distribution that is causing it, so even if we wanted to, we could not say we were sitting at some fixed distance from the source. In that case, we could not even tell what strength the gravitational field had, i.e., the 'acceleration due to gravity' could be anything at all. This is discussed at length in [5].

Steane says that the notion of what is observed by observers fixed in the whatever frame we have is better captured by 'including' the metric in the definition. Of course, what we are doing is including the metric in the definition of the force per unit (stationary) charge via the Ori–Rosenthal fix, and still calling this the electric field, viz.,

$$E^i := (-g_{00})^{1/2} F^i{}_\lambda u^\lambda = F^i{}_0 \ .$$

This move doesn't sound unreasonable if we are in the game of defining physical quantities 'for' accelerating observers. But the point is that it doesn't matter how we define physical quantities, provided that we can predict what will happen in experiments. ‘Pictures’ for observers, of any kind, even inertial, are irrelevant.

And worse, for accelerating observers there aren’t even any physically canonical pictures. What seems awkward with accelerating observers is this: how should we interpret quantities expressed relative to non-inertial coordinate systems (even if they are Killing observers, like the Rindler observer)? Are we interested in what accelerating observers actually measure, or are we just trying to make good definitions for them. But what will accelerating observers observe? What will they consider to be good definitions? And if they are good, what are they good for? What exactly are we trying to achieve? What will accelerating observers measure using accelerating detectors? Indeed, does it help to know what accelerating detectors will detect?

After all, there is a major theoretical difference between inertial motion and accelerating motion, both for observers and for detectors. When an observer is moving inertially, we know what are the best coordinates for such a person to use: they are inertial or locally inertial coordinates. This is because all our field theories of matter are Lorentz symmetric or locally Lorentz symmetric, and these are the coordinate systems in which they assume their simplest forms.

Regarding detectors, imagine designing two different detectors to measure the same physical quantity. Whenever they are moving inertially in the same physical context, we expect them to deliver the same value for whatever quantity it is they are supposed to measure. This is once again because all our field theories of matter, which govern both the internal constitution of the detectors and the environment of the detectors, are Lorentz symmetric or locally Lorentz symmetric. But what can we say when they are accelerating? Will they always deliver the same result for the given physical quantity? After all, there is no corresponding acceleration symmetry in our field theories of matter.

There appears to be no attempt to consider what accelerating observers might really do in practice. But neither is there any theoretical justification. Why do we have an Ori–Rosenthal type of force which includes a factor of $\mathrm{d}\tau/\mathrm{d}t$? What equation of motion does it appear in? The same question should be addressed for the self-force defined according to (49) on p. 30. Why should that be physically relevant? Of course, in each case, one can show that it appears in an equation of motion of sorts for these charge distributions, which tells us how the object as a whole will move, and how we might take self-force effects into account in its effective inertial mass. These are the only crucial issues here.

12 Conclusion

The conjecture in [7] to the effect that the leading order term in the EM self-force of a spatially extended charge distribution is always aligned with the four-acceleration of the distribution was not correct. This is sometimes true, however, and in particular in the case of a spherically symmetric charge distribution with arbitrary acceleration, and a charge dumbbell accelerated either perpendicular to its axis or along its axis.

Following Steane, who has considered the case of a charge dipole with uniform acceleration [10], we have shown how the unmentioned force that prevents the dumbbell from collapsing (when e_A and e_B have opposite signs) or falling apart (when they have the same sign) will give rise to a self-force which exactly cancels the unwanted term in the EM self-force on a charge dumbbell with arbitrary acceleration, so that the total self-force is in fact always aligned or counteraligned with the acceleration.

Indeed, if this balancing force, the Poincaré stress (PS), arises due to some field for which the two entities A and B are sources, a self-force effect can be expected due to this field. It is important to remember that the model of the spatially extended particle is not physically complete without considering all the relevant forces. In the present model, we imagine the two charges to be connected by a massless rod containing a massless fluid and solve the relativistic Navier–Stokes equation (31) for the pressure in the fluid. This allows us to calculate what the PS self-force on the system must be under the constraint of rigid motion.

The rogue term in the general formula (27) for the EM self-force on the charge dumbbell, the term non-aligned with the acceleration, contributes for longitudinal motion. This rogue term is paying for Steane's hidden momentum (see Appendix C), and not just in the case of longitudinal motion. If Poincaré stresses introduce a self-force that corrects the total longitudinal self-force to the total transverse self-force, as suggested in [7], then one expects them also to do so for arbitrary orientations.

If we consider something like the proton, which involves strong forces as well as EM forces, it may seem surprising that such a crude model could deliver the self-force due to all forces other than the EM force. For of course, any strong self-force (if one could calculate it as such) would be much greater than the EM self-force. So how could the above pressure model for the Poincaré stresses solve the problem of the non-aligned term without knowing anything about the strong force?

The answer is that rigid motion is a very strong constraint, and a rather static one to say the least, while the proton is a dynamic equilibrium. What we analyse here are very crude models, nothing like the QCD models of the proton and the like in [3], where the light hadron masses are actually derived from structural simulations. But it is still striking that classical electromagnetism is not renormalisable in the point particle limit, while a 'complete' (i.e., stabilised) model involving all other relevant forces would be.

So Poincaré stresses can correct the EM self-force in this way without our having any knowledge whatever of their nature, e.g., even if the particle is a proton and they are due to the strong force! This comes about precisely because rigid motion is such a stringent constraint. Indeed, it is a constraint that suffices to deliver the self-forces due to all other fields for which the particles are sources. We just don't need any other details about those fields *if* they are capable of ensuring rigid motion. But naturally, the innards of a proton will *never* satisfy the rigid motion condition. Here we would have a much more dynamic equilibrium.

Poincaré originally introduced his so-called stresses to resolve the discrepancy between the energy-derived and momentum-derived masses of the fields [7, Chap. 11] (see also the note on pp. 4–5 of [9]). What we see here is that they also resolve the discrepancy between the energy-derived and *self-force-derived* masses of the fields, which is not entirely surprising since momentum gets into the fields by doing work against the leading order (potentially divergent) term in the self-force, and we know that the momentum-derived and self-force-derived masses of the fields are always equal [7, Chap. 3].

In appendix we provide plausible arguments for extending all of the above analysis to arbitrary continuous charge distributions in arbitrary rigid motion. However, we note that the 'pressure' model of the Poincaré stresses becomes somewhat strained, in the sense that it would be hard to imagine its physical implementation. Here it would be useful to find a more elegant formulation of these effects.

We have discussed at length Steane's pictures for observers supported in static homogeneous gravitational fields (SHGFs). He shows that an observer at rest in an SHGF (he means sitting fixed at the space origin of some Rindler coordinates), described everywhere by the Rindler metric (which is of course the Minkowski metric expressed relative to Rindler coordinates), will find any charged object supported in the field (he means sitting at some fixed Rindler space coordinates) to possess an EM self-force equal to that observed in an inertial frame when the same object moves with constant acceleration and fixed proper size and shape. We explain how this is intended to be a statement of the accidental equality (53) on p. 32 between the expressions for EM four-forces expressed relative to Rindler coordinates or a suitable inertial frame. We note, however, that if the Rindler coordinates are intended to represent an SHGF, we do require the strong equivalence principle (SEP) to ship Maxwell's equations into the GR context.

We question the need for pictures adapted to accelerating observers on several grounds, noting in particular that there is no attempt to determine what such people might actually do in order to establish such pictures, whence this would appear to be a purely theoretical exercise. The present view is that we need only predict what will happen to physical systems in different contexts. There can be no deep physical significance in the suggested pictures.

Hence, redefinitions of physical quantities like the electric field or four-forces 'for' accelerating observers is a dubious exercise from a physical point of view. But Steane does identify an important theoretical issue regarding the calculation of self-forces on extended objects, viz., the problem that *no* definition can be covariant in general, because any definition must depend on a foliation of the spacetime. He considers one foliation to be more suitable to inertial observers and another to be more suitable to accelerating observers, and in particular, observers 'supported' in an SHGF. He shows how one can reach qualitatively identical conclusions about the alignment of the total self-force in the two cases, and numerically identical conclusions about the value of the self-force-derived mass renormalisation.

But it is not clear from Steane's analysis why this should work out, given the redefinition of the self-force, and it is not clear whether it would always work out for arbitrary charge distributions with arbitrary accelerations. Further, it is very unlikely that all the quantitative details will work out to be the same in the two ‘pictures’. In fact, there is no obvious need to make a picture for any accelerating observer. It is suggested here that a better approach is to examine the equation of motion of the charge distribution for one of the definitions of the self-force and thereby derive its motion. This will be the same whichever picture we use, so there would then be no need to redefine the self-force, examine the corresponding equation of motion, and rederive the same motion in another way.

In their paper, Ori and Rosenthal [9] show that one may use the alternative definition of the EM self-force to obtain an equation of motion for the charge dumbbell in the point particle limit that is entirely consistent with the usual definition. Their account is described in detail in appendix. Here we point out with Steane [10] that the self-force due to internal forces (Poincaré stresses) must be zero when we use this same altered definition for sums of forces acting at different spacetime events, and we show by generalising a method due to Steane [10] that this will indeed be the case, provided that the motion of the dumbbell is rigid. We also suggest that this whole method depends heavily on the rigidity of the motion and would not be generalisable to ‘particles’ whose internal structure involves a dynamic equilibrium, such as the proton.

We argue against Steane’s proposal [10] that Ori and Rosenthal’s method is merely a natural choice for a certain kind of observer, namely one comoving with the dumbbell. What matters here is the equation of motion of the system and its resulting worldtube. In the present case, when the various assumptions are satisfied and the approximations justified, we obtain exactly the same worldline in the point particle limit as we would with the previous method, so this clearly constitutes a consistent reanalysis of the problem that has nothing to do with any observer. Outside the point particle limit, there are likely to be some small differences between the predictions due to the different approximations.

The point about equations of motion is that they give us something that is independent of any coordinate system, namely a worldline or a worldtube. It thus has an immediate physical significance. Pictures for observers are by definition coordinate dependent, and if there is one major lesson from the relativity theories, and in particular from the general theory of relativity, it is that coordinate particularities of things are physically irrelevant. Naturally, if one is doing a sky survey, it is important to be able to set up frames of reference, and for this one must make operational definitions and assumptions. However, the pictures here seem to be motivated largely by some theoretical urge and there is no mention of the actual practical problems, whence they appear worthless.

A Ori–Rosenthal Approach

The paper [9] provides a rather elegant presentation of an alternative way of deriving self-force effects which does not require any adjustment due to Poincaré stresses in order to obtain a leading order term in the EM self-force that can be removed by mass renormalisation. As explained by Steane [10], it does indeed advocate a different way to sum forces acting at different spacetime events, but their aim is not to advocate one way for one kind of observer and another way for another kind of observer. Their intention is to show *directly* that the relativistic version of Newton's second law can be rescued in the point particle limit (for a charge dumbbell) by absorbing potentially divergent self-force terms into the inertial mass of the system (times the acceleration).

For convenience, we relay their formulation of the dumbbell motion, since it is short and sweet by comparison with my own in Sect. 3, although somewhat less didactic.

A.1 Dumbbell Structure and Kinematics

We have two electric charges q_+ and q_- at the ends of a rod of proper length 2ε. The charges may differ in magnitude and sign. The total charge is $q := q_+ + q_-$. The proper length is assumed time independent and the motion as non-rotational as possible, i.e., we have rigid motion and the rod axis is Fermi–Walker (FW) transported along the worldline of either charge.

The central point of the rod is taken to represent the dumbbell's motion. It has worldline $z^\mu(\tau)$, where τ is the proper time. The four-velocity and four-acceleration of the central reference point are $u^\mu := \dot{z}^\mu$ and $a^\mu := \dot{u}^\mu$. The worldlines of the charges are denoted by $z^\mu_\pm(\tau)$, expressed as functions of τ. One would expect them to be functions of the proper times $\tau_\pm$ of the charges, but these can be turned into functions of τ. The point is that, for all τ, we can look for the intersection of the worldlines of each charge with the hyperplane of simultaneity in the instantaneous rest frame of the central point and we can uniquely associate a τ_+ and a τ_- with each given τ.

We now define a unit spacelike vector $w^\mu(\tau)$ for each τ by the relations

$$z_\pm(\tau) = z(\tau) \pm \varepsilon w(\tau) \ , \qquad w_\mu w^\mu = 1 \ , \qquad w_\mu u^\mu = 0 \ . \tag{58}$$

We also insist that w is FW transported along the worldline $z(\tau)$. FW transport of w along z is expressed by

$$\dot{w}^\mu = (u^\mu a^\nu - u^\nu a^\mu) w_\nu \ ,$$

or without the indices,

$$\dot{w} = u(a \cdot w) - a(u \cdot w) \ .$$

We define

$$a_w := a_\lambda w^\lambda = a \cdot w \ ,$$

and given that we hope always to have $u \cdot w = 0$, the equation of motion for w becomes

$$\dot{w} = a_w u = (a \cdot w) u \ . \tag{59}$$

Of course, FW transport ensures that, if we can arrange for $w \cdot w = 1$ and $w \cdot u = 0$ at some initial time, then they will hold for all time. This is immediate from

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(w \cdot w) = 2w \cdot \dot{w} = 2w \cdot \Big[u(a \cdot w) - a(u \cdot w) \Big] = 0 \ ,$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}(w \cdot u) = u \cdot \dot{w} + w \cdot \dot{u} &= u \cdot \Big[u(a \cdot w) - a(u \cdot w) \Big] + w \cdot a \\ &= -(u \cdot w)(u \cdot a) \qquad (\text{as } u^2 = -1) \\ &= 0 \qquad (\text{as } u \cdot a = 0) \ . \end{aligned}$$

So the whole scenario ends up being logically consistent.

The four-velocities of the charges are

$$u_\pm := \frac{\mathrm{d}z_\pm}{\mathrm{d}\tau_\pm} = \frac{\mathrm{d}}{\mathrm{d}\tau_\pm}(z \pm \varepsilon w) = \frac{\mathrm{d}}{\mathrm{d}\tau}(z \pm \varepsilon w)\frac{\mathrm{d}\tau}{\mathrm{d}\tau_\pm}$$
$$= (u \pm \varepsilon \dot{w})\frac{\mathrm{d}\tau}{\mathrm{d}\tau_\pm}$$
$$= u(1 \pm \varepsilon a_w)\frac{\mathrm{d}\tau}{\mathrm{d}\tau_\pm} \qquad \text{[by (59)]} \; . \tag{60}$$

Since u and $u_\pm$ all 'square' to -1, we have

$$-1 = -(1 \pm \varepsilon a_w)^2 \left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau_\pm}\right)^2 \, ,$$

and hence the key equation

$$\frac{\mathrm{d}\tau_\pm}{\mathrm{d}\tau} = 1 \pm \varepsilon a_w \; . \tag{61}$$

Of course, we know we should take the positive square root for small ε. Going back to (60), we deduce that

$$u_\pm = u \; . \tag{62}$$

This proves the existence of the ICIF for the whole system at any given time τ. However, this alone does not establish rigidity of the motion. For that, one must show that the proper distance between the endpoints is constant *in the instantaneous rest frame of either of them*. The point is that the displacement between the endpoints is the spacelike vector

$$z_+ - z_- = 2\varepsilon w \, ,$$

and

$$(z_+ - z_-)^2 = 4\varepsilon^2 \, ,$$

whence $|z_+ - z_-| = 2\varepsilon$ is a constant in time. We just have to note here that $z_+ - z_-$ is orthogonal to $u = u_\pm$, hence lies entirely in the hyperplane of simultaneity of the ICIF. Rigidity was thus effectively decreed in (58), but Fermi–Walker transport of the rod axis plays a crucial role.

The four-accelerations of the charges are

$$a_\pm := \frac{\mathrm{d}u_\pm}{\mathrm{d}\tau_\pm} = \frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\tau_\pm} = \frac{a}{1 \pm \varepsilon a_w} \, , \tag{63}$$

by (61).

A.2 Mutual Forces

Since we are interested in the EM force that each charge exerts on the other, the first thing is to use Dirac's result (7) back on p. 8, viz.,

$$F^{\mu\nu} = q\big[1 - (\gamma \cdot \dot{v})\big]^{-1/2}\bigg\{ -\epsilon^{-3} v^\mu \gamma^\nu - \frac{1}{2}\epsilon^{-1}\dot{v}^\mu v^\nu \big[1 + (\gamma \cdot \dot{v})\big] \tag{64}$$
$$+ \frac{1}{8}\epsilon^{-1}\dot{v}^2 v^\mu \gamma^\nu + \frac{1}{2}\epsilon^{-1}\ddot{v}^\mu \gamma^\nu + \frac{2}{3}\dddot{v}^\mu v^\nu - (\mu \longleftrightarrow \nu)\bigg\} \, ,$$

to get the retarded EM field tensor that a single point charge q moving on an arbitrary worldline $z(\tau)$ produces at a nearby point $z + \hat{\varepsilon}\hat{w}$, where $\hat{\varepsilon}$ is a small positive number ($\hat{\varepsilon} = 2\varepsilon$) and $\hat{w}$ is a unit spacelike vector satisfying $\hat{w}\cdot\hat{w} = 1$ and $\hat{w}\cdot u = 0$. Note that (64) is accurate to zero order in ϵ, so we will obtain this tensor to the same order in $\hat{\varepsilon}$.

We have the correspondence

$$\epsilon \longleftrightarrow \hat{\varepsilon}\ , \qquad v \longleftrightarrow u\ , \qquad \gamma \longleftrightarrow \hat{\varepsilon}\hat{w}\ , \qquad \dot{v} \longleftrightarrow a\ , \qquad \ddot{v} \longleftrightarrow \dot{a}\ .$$

The oppositie sign conventions for the metric also mean that

$$-\gamma\cdot\dot{v} \longleftrightarrow +\hat{\varepsilon}\hat{w}\cdot a = \hat{\varepsilon}a_{\hat{w}}\ , \qquad \dot{v}^2 \longleftrightarrow -a^2\ .$$

Feeding this in, we soon obtain the first line of Ori and Rosenthal's formula (10) on p. 12, viz.,

$$F_{\mu\nu} \approx \frac{q}{\sqrt{1+\hat{\varepsilon}a_{\hat{w}}}}\left[\frac{u_\mu\hat{w}_\nu}{\hat{\varepsilon}^2} + \frac{a_\mu u_\nu}{2\hat{\varepsilon}} + \frac{a^2u_\mu\hat{w}_\nu}{8} - \frac{\dot{a}_\mu\hat{w}_\nu}{2} - \frac{a_{\hat{w}}a_\mu u_\nu}{2} - \frac{2}{3}\dot{a}_\mu u_\nu - (\mu\longleftrightarrow\nu)\right],$$

where the approximation sign indicates in the following that we have dropped terms of order $\hat{\varepsilon}$ and higher.

We can also expand out the initial multiplicative factor using the expansion

$$f(x) := (1+x)^{-1/2} \quad\Longrightarrow\quad f(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + O(x^3)\ ,$$

whence

$$(1+\hat{\varepsilon}a_{\hat{w}})^{-1/2} = 1 - \frac{1}{2}\hat{\varepsilon}a_{\hat{w}} + \frac{3}{8}\hat{\varepsilon}^2a_{\hat{w}}^2 + O(\hat{\varepsilon}^3)\ .$$

This soon gives the second line of Ori and Rosenthal's formula (10) on p. 12, viz.,

$$F_{\mu\nu} \approx q\left[\frac{u_\mu\hat{w}_\nu}{\hat{\varepsilon}^2} + \frac{a_\mu u_\nu - a_{\hat{w}}u_\mu\hat{w}_\nu}{2\hat{\varepsilon}} - \frac{2}{3}\dot{a}_\mu u_\nu + \hat{Z}_{\mu\nu} - (\mu\longleftrightarrow\nu)\right]\ , \qquad (65)$$

where we isolate the terms

$$\hat{Z}_{\mu\nu} := \frac{a^2u_\mu\hat{w}_\nu}{8} - \frac{\dot{a}_\mu\hat{w}_\nu}{2} + \frac{3a_{\hat{w}}^2u_\mu\hat{w}_\nu}{8} - \frac{3a_{\hat{w}}a_\mu u_\nu}{4}\ ,$$

because, apart from being $O(\hat{\varepsilon}^0)$, they are odd order in $\hat{w}$ and will eventually drop out of the calculation.

Let $F_+^{\mu\nu}$ be the field produced by the charge q_- at the charge q_+, obtained by the substitutions

$$q \longrightarrow q_-\ , \quad a \longrightarrow a_-\ , \quad \dot{a} \longrightarrow \frac{\mathrm{d}a_-}{\mathrm{d}\tau_-}\ , \quad \hat{w} \longrightarrow w\ , \quad a_{\hat{w}} \longrightarrow a_-^\lambda w_\lambda\ , \quad \hat{\varepsilon} \longrightarrow 2\varepsilon\ ,$$

while $F_-^{\mu\nu}$ be the field produced by the charge q_+ at the charge q_-, obtained by the substitutions

$$q \longrightarrow q_+\ , \quad a \longrightarrow a_+\ , \quad \dot{a} \longrightarrow \frac{\mathrm{d}a_+}{\mathrm{d}\tau_+}\ , \quad \hat{w} \longrightarrow -w\ , \quad a_{\hat{w}} \longrightarrow -a_+^\lambda w_\lambda\ , \quad \hat{\varepsilon} \longrightarrow 2\varepsilon\ ,$$

noting that we can put u for $u_\pm$, thanks to (62). The Lorentz force f_+ that the charge q_- exerts on the charge q_+ and the Lorentz force f_- that the charge q_+ exerts on the charge q_-, defined as usual by

$$f_\pm^\mu := q_\pm F_\pm^{\mu\nu}u_\nu\ ,$$

are then given by Ori and Rosenthal's equation (12) on p. 12, viz.,

$$f^\mu_\pm \approx q_+q_-u_\nu\left[\pm\frac{u^\mu w^\nu}{4\varepsilon^2}+\frac{a^\mu_\mp u^\nu-(a^\lambda_\mp w_\lambda)u^\mu w^\nu}{4\varepsilon}-\frac{2}{3}\dot{a}^\mu_\mp u^\nu\pm Z^{\mu\nu}_\pm-(\mu\longrightarrow\nu)\right], \quad (66)$$

where we have defined

$$Z^{\mu\nu}_\pm := \frac{a^2_\mp u^\mu w^\nu}{8}-\frac{\dot{a}^\mu_\mp w^\nu}{2}+\frac{3(a^\lambda_\mp w_\lambda)^2u^\mu w^\nu}{8}-\frac{3(a^\lambda_\mp w_\lambda)a^\mu_\mp u^\nu}{4}\;,$$

and

$$\dot{a}_\pm := \frac{\mathrm{d}a_\pm}{\mathrm{d}\tau_\pm}\;.$$

The next step is to express $f_\pm$ in terms of the acceleration a and proper time τ of the central point of the dumbbell, rather than those of the charges themselves. We thus expand (63) in powers of ε. The acceleration does not appear in the $O(\varepsilon^2)$ term of $f_\pm$, so we only need the expansion to order ε, viz.,

$$a_\mp = a(1\pm\varepsilon a_w)+O(\varepsilon^2)\;.$$

Since $\dot{a}_\mp$ only appears in the $O(\varepsilon^0)$ term, it can be replaced directly by $\dot{a}$, and the same goes for all appearances of $a_\pm$ and $a^2_\pm$ in the $O(\varepsilon^0)$ term. The result is

$$f^\mu_\pm \approx q_+q_-u_\nu\left[\pm\frac{u^\mu w^\nu}{4\varepsilon^2}+\frac{a^\mu u^\nu-a_w u^\mu w^\nu}{4\varepsilon}-\frac{2}{3}\dot{a}^\mu u^\nu\pm Z^{\mu\nu}-(\mu\longrightarrow\nu)\right]\;, \quad (67)$$

with

$$Z^{\mu\nu} := Z^{\mu\nu}_\pm(a_\pm\to a)+\frac{a_w a^\mu u^\nu-a^2_w u^\mu w^\nu}{4}\;.$$

The second term in the latter is essentially all that is new after putting $a_\pm\to a$ in (66). We observe that the $O(\varepsilon^0)$ term $Z^{\mu\nu}$ appears with opposite signs for the two Lorentz forces, as does the Coulomb term which goes as ε^{-2}. These would drop out of a simple sum of f_- and f_+.

Now using

$$u^\nu u_\nu=-1\;,\qquad w^\nu u_\nu=0=a^\nu u_\nu\;,\qquad \dot{a}^\nu u_\nu=-a^2\;,$$

where the latter is obtained by differentiating the identity $a^\nu u_\nu=0$, we get Ori and Rosenthal's equation (14) on p. 13, viz.,

$$f^\mu_\pm \approx q_+q_-\left[\pm\frac{w^\mu}{4\varepsilon^2}-\frac{a^\mu+w^\mu a_w}{4\varepsilon}+\frac{2}{3}(\dot{a}^\mu-a^2u^\mu)\pm Z^\mu\right]\;, \quad (68)$$

with

$$Z^\mu := u_\nu(Z^{\mu\nu}-Z^{\nu\mu})\;.$$

If we add f_- and f_+ together, as we were doing in Sect. 4 and as Steane does in the first half of his paper [10] (for the special case of uniform acceleration), evaluating them at times τ_- and τ_+ that correspond to the same ICIF, what we get is

$$f_{\mathrm{sum}}=f_-+f_+\approx-\frac{q_-q_+}{2\varepsilon}(a+wa_w)+\frac{4}{3}q_-q_+(\dot{a}-a^2u)\;. \quad (69)$$

Ori and Rosenthal's problem with this is that the first term on the right-hand side is proportional to ε^{-1} and so diverges in the point particle limit, while the usual process of mass renormalisation will not work because the term to be removed has to be proportional to a. But clearly, the rogue term goes as $a + wa_w$. Not only is this not aligned with a, but it depends on the orientation of the dumbbell.

Steane has solved this problem for the uniform acceleration case [10] and the present paper solves it for the general case considered by Ori and Rosenthal (see Sect. 7). In brief, there have to be Poincaré stresses within the rod, modelled here by a 'pressure', in order to ensure its stability, and in particular to ensure that it can actually undergo *rigid* motion. But there must then be a pressure gradient along the dumbbell axis and this will end up exerting its own self-force. The latter precisely removes the unwanted part wa_w of the rogue term.

But Ori and Rosenthal have another solution here which we will argue in the following is entirely consistent from a purely physical point of view. It does indeed redefine the summing of forces acting at different spacetime events, and in precisely the way spelt out by Steane [10]. But we will argue, in contradistinction to Steane, that there is no need to wait upon this or that observer before calling upon their method.

A.3 Energy–Momentum Balance: A Different Sum

We denote the dumbbell four-momentum by $p^\mu(\tau)$ at any proper time τ of the central point. This is defined to be the integral of the appropriate components of the dumbbell stress–energy tensor over the ICIF σ which we know always to exist for rigid motion. Now $u(\tau)$ is orthogonal to $\sigma(\tau)$, so if $\mathrm{d}^3\sigma$ is the volume element in σ, we define

$$p^\mu := -\int_\sigma T^{\mu\nu}_{\mathrm{dumb}} u_\nu \mathrm{d}^3\sigma \ . \tag{70}$$

We should note several things about this definition:

- The first point is that T_{dumb} is the stress–energy tensor for the dumbbell *without* EM fields, so it is zero outside the spacetime region occupied by the dumbbell. According to the authors, this exclusion of EM fields is motivated by the fact that the EM contribution is not well localised, being partly scattered through space in the form of EM waves. Because the non-EM part of the four-momentum of the system is well localised, monitoring $p(\tau)$ provides the information we need about the dumbbell motion. This will be shown explicitly in a moment.
- Second, the integral is carried out over the hyperplane of simultaneity of the ICIF and not over a hyperplane of simultaneity of some arbitrary inertial frame. As Steane explains, the definition of any total four-momentum will generally depend on the choice of spacelike hypersurface (see Sect. 9).
- Third, Ori and Rosenthal describe this as the *natural covariant way* to define the time-dependent four-momentum of a rigid body.

But is it covariant? In [7, Chap. 11], there is a long discussion to the effect that one cannot obtain a Lorentz covariant four-vector by integrating a non-conserved stress–energy tensor over spacelike hypersurfaces. So what is different here?

The answer is that we have *decreed* the hypersurface in question to be the hyperplane of simultaneity (HOS) of the instantaneous rest frame, and because it is specified physically by the system itself, it will not be affected by a change of inertial frame. What goes wrong when we use the HOS of some otherwise arbitrary inertial frame is that the HOS itself changes when we change frame. And of course, p defined by (70) is *obviously* covariant, i.e., it will transform as a four-vector under change of inertial frame.

As discussed in [7, Chap. 11], this was how Rohrlich redefined the total four-momentum of the EM fields around a charged spherical shell in order to obtain a covariant quantity and hence 'solve' the notorious 4/3 problem in that case. It was argued there that it was not a physically enlightening move, in part because it is a rather cheap and artificial way to get a Lorentz covariant quantity. But here we have effectively the same ploy again and its authors consider it to be the most *natural* definition.

In the present case, however, it will turn out that we can make a physical case for following this path, although only in a very special case. A key limitation is this: the method depends heavily on there always being an ICIF for the whole system, and this in turn depends on the system being capable somehow of rigid motion. So this idea is *not* generalisable. The same problem was raised with regard to Rohrlich's ploy (see [7, Sect. 11.4]).

What about other forces? We expect there to be some external force acting on the system to ensure that it has the rigid acceleration attributed to it through the functions $z(\tau)$ and $z_{\pm}(\tau_{\pm})$. And we should ask also whether there are no Poincaré stresses, i.e., internal forces apart from those due to the retarded Lienard–Wiechert fields that each charge produces at the location of the other, as might be required to ensure the stability of the system and allow it to move rigidly. It is interesting to see what Ori and Rosenthal say about that, since it was those very forces that saved the other way of renormalising the equation of motion.

Their position is stated on p. 9 of their paper [9]. They begin by noting, as is well known, that Newton's third law of action and reaction does not apply to EM forces. This failure may be attributed to the long range of the electromagnetic interactions between charges, which is itself responsible for radiative effects transporting energy and momentum away from the interacting charges. In contrast, they assume that any internal forces will be short range, whence the sum of such forces will always cancel out *except for a mass-renormalisation-like term*. They attempt further justification in a footnote: the range of the internal forces will be small compared to the dumbbell length, something one would expect for the interatomic forces responsible for the rigidity of the rod.

This assumption effectively does away with any hope of a Poincaré stress (PS) self-force correcting any misalignment of the leading order term in the EM self-force. On the other hand, it is not obvious how their justification really works, since we know that, under the conditions of the calculation in Sect. 4 of the present paper, and in the first part of Steane's paper [10], one cannot just decree that the sum of internal forces will cancel out, or that it will only contribute a mass-renormalisation-like term which must a fortiori be aligned with the dumbbell acceleration. Under those conditions, the Poincaré stresses provide a term that is *not* mass-renormalisation-like because it points along the dumbbell axis.

But then we know from Steane's calculation of the PS self-force *in the Ori–Rosenthal way* (see the discussion in Sect. 10.2 and in particular p. 38) that, at least in the case of uniform acceleration, there will be no such mass renormalisation, and indeed no PS self-force. And this calculation, using the pressure model again, only requires one to know that the motion of the system is rigid, whence the accelerations of imaginary particles along the dumbbell axis must be distributed in a certain well defined manner given by (57), combined with the fact that the proper time factors are distributed according to (55) on p. 37, which is another direct consequence of the rigidity assumption.

And we can repeat all this for the case of arbitrary acceleration as formulated in this paper and in Ori and Rosenthal's paper. This is perhaps an opportune moment to do that. We have the pressure formula (39) on p. 23, viz.,

$$p(s) = \frac{k}{1 + s\mathbf{d} \cdot \mathbf{a}_0} \ , \tag{71}$$

where k is a constant, and this is directly analogous to (57). We have to multiply $p(s)$ by $\mathrm{d}\tau_s/\mathrm{d}\tau_{\mathrm{c}}$ as given by (34) on p. 21, viz.,

$$\frac{\mathrm{d}\tau_s}{\mathrm{d}\tau_A} = 1 + s\xi^i a_{0i} = 1 + s\mathbf{d} \cdot \mathbf{a}_0 \ , \tag{72}$$

taking the reference point to be A. But the product is just the constant k.

So it looks like we can actually prove that, for rigid motion, the Ori–Rosenthal kind of total force cannot have a PS component. But it should be noted in the following that they do not consider their approach to be a picture 'for' the accelerating observer comoving with the dumbbell, and nor do they really view it as a redefinition, but rather as a reanalysis. We shall see shortly that they do justify their equation of motion derived from the definition (70) of the dumbbell four-momentum p, *under a certain assumption* which once again amounts to the rigidity assumption.

Let us see how they proceed here. In the name of energy–momentum conservation, they deduce that $p(\tau)$ will only change due to external forces acting on the dumbbell and due to energy–momentum exchange between the dumbbell and the EM field. Since we have just disposed of the PS self-force due to internal forces, this seems fair enough. The EM energy–momentum exchange is manifested by the EM forces acting on the two charges, and in a time interval $\mathrm{d}\tau$, the change in $p(\tau)$ will thus be

$$\mathrm{d}p = \mathrm{d}p_+ + \mathrm{d}p_- + \mathrm{d}p_{\mathrm{ext}} \, , \tag{73}$$

where $\mathrm{d}p_{\mathrm{ext}}$ is the contribution from the external force and $\mathrm{d}p_\pm$ are the contributions from the EM forces acting on the two charges due to the fields they themselves produce. If the PS self-force were not zero, we should require another term $\mathrm{d}p_{\mathrm{PS}}$ here, so this is where the authors explicitly assume that it will not contribute, i.e., that the 'pressure' in the dumbbell rod can do no work.

If we denote the EM forces just mentioned, i.e., those acting on the two charges due to the fields they themselves produce, by $f_\pm^{\mathrm{em}}$, we can write

$$f_\pm^{\mathrm{em}} = f_\pm + \hat{f}_\pm \, ,$$

where $f_\pm$ are given by (68) and $\hat{f}_\pm$ are what Ori and Rosenthal refer to as the partial self-forces acting on the charges $q_\pm$, i.e., the self-forces they exert *after mass renormalisation*, in the point particle limit of whatever spatially extended charge distributions we take to model them. Note that these authors use the term 'EM self-force' to refer only to the radiative term that remains after the point particle limit has been taken, so $\hat{f}_\pm$ are the radiative force terms due to the motions of the charges $q_\pm$.

This is another important part of their argument and looks cogent. They note that the radiative self-force is found to be universal in the small size limit. Indeed, it is independent of the size by definition, and also independent of the shape and orientation of the object. We thus expect it to be there for any sufficiently small charged object, such as the two charges q_+ and q_-. This fact can be used to relate $\hat{f}_\pm$ to the overall EM self-force acting on the dumbbell (see below).

So we have (73) and the next task is to calculate $\mathrm{d}p_\pm$, the energy–momentum exchange of the $q_\pm$ charge with the EM field, between the two hypersurfaces of simultaneity parametrised by τ and $\tau + \mathrm{d}\tau$. The proper time elapsed at the charge $q_\pm$ between these two hypersurfaces is $\mathrm{d}\tau_\pm$, suggesting that the amount of EM energy–momentum transfer here will be

$$\mathrm{d}p_\pm = f_\pm^{\mathrm{em}} \mathrm{d}\tau_\pm \, .$$

We thus obtain Ori and Rosenthal's equation (20), viz.,

$$\mathrm{d}p = (f_+ + \hat{f}_+)\mathrm{d}\tau_+ + (f_- + \hat{f}_-)\mathrm{d}\tau_- + \mathrm{d}p_{\mathrm{ext}} \, . \tag{74}$$

There is a physical assumption here that any change in what we have defined to be p in (70), between the HOSs of the two ICIFs, must be equal to the sum of the changes in $p_\pm$ as calculated in this way, plus the contribution from the external force. The critical point is to see that this is just an expression of energy–momentum conservation. Once we accept that, we are basically through.

We define the overall force f acting on the system to be

$$f := \frac{\mathrm{d}p}{\mathrm{d}\tau} \, , \tag{75}$$

then divide both sides of (74) by $\mathrm{d}\tau$ to obtain

$$f \approx f_+ \frac{\mathrm{d}\tau_+}{\mathrm{d}\tau} + f_- \frac{\mathrm{d}\tau_-}{\mathrm{d}\tau} + \hat{f}_+ + \hat{f}_- + f_{\mathrm{ext}} \, . \tag{76}$$

As before, the approximation sign indicates that we have dropped some terms going as ε. The point is that $\hat{f}_\pm$ are expected to be regular when $\varepsilon \to 0$, and since

$$\frac{\mathrm{d}\tau_\pm}{\mathrm{d}\tau} = 1 \pm \varepsilon a_w = 1 + O(\varepsilon) ,$$

according to (61) on p. 48, we can write

$$\hat{f}_\pm \frac{\mathrm{d}\tau_\pm}{\mathrm{d}\tau} = \hat{f}_\pm + O(\varepsilon) .$$

Something similar happens with f_{ext}. On the face of things it looks as though we have assumed that the external force is applied at only one point of the dumbbell, but this is not a necessary assumption. We could replace it by a sum of external forces $\sum f_{\mathrm{ext}}$ acting at different points, and the fact that each term of $\sum f_{\mathrm{ext}}$ is likely to be regular in the limit $\varepsilon \to 0$ means that the proper time factor $\mathrm{d}\tau_\pm/\mathrm{d}\tau$ will not vary sufficiently from one end of the dumbbell to the other to make any difference to $O(\varepsilon)$. In brief, f_{ext} can be taken to denote a sum of forces acting at different points of the dumbbell.

So it is only the mutual force $f_\pm$ that requires us to make the distinction between $\mathrm{d}\tau$ and $\mathrm{d}\tau_\pm$, simply because it contains a term with a negative power of ε. We define the overall mutual EM force f_{mutual} in the obvious way by

$$f_{\mathrm{mutual}} := f_+ \frac{\mathrm{d}\tau_+}{\mathrm{d}\tau} + f_- \frac{\mathrm{d}\tau_-}{\mathrm{d}\tau} . \tag{77}$$

This is indeed Steane's second way of defining a sum of forces acting at different spacetime events (see Sect. 9). Of course, this is a definition, perhaps a different definition from something we used previously, but we have an expression for it from all the above theory (see below), so the key physical problem is to know what we mean by the left-hand side of (76). So far, it is given by (75), which is just a definition of 'total force', but this too will soon be understood physically.

There is one more point about (76) before we write out our expression for f_{mutual}. If we had had a term $\mathrm{d}p_{\mathrm{PS}}$ in (74), possibly distributed along the dumbbell axis like the pressure (71) in our argument above, we can see why we would have had to include the distributed factor (72) along the dumbbell and end up with a constant that would not have affected the change in the total energy–momentum.

This could be formulated more precisely by imagining the internal forces as distributed along the dumbbell axis, just as we imagined the external forces might be a moment ago, proportional to something like the expression in (71) at parameter distance s along the axis, but multiplied there by something like the expression in (72), whence we obtain a constant which will be the same from one ICIF to the next. As explained on p. 52, this only depends on the internal forces doing what is required to maintain rigidity of the dumbbell throughout its motion, provided that the pressure model is a good one.

So what is the expression for f_{mutual}? By (61) on p. 48 and (68) on p. 50, we have

$$\begin{aligned} f_\pm \frac{\mathrm{d}\tau_\pm}{\mathrm{d}\tau} &= (1 \pm \varepsilon a_w) f_\pm \\ &\approx q_+ q_- (1 \pm \varepsilon a_w) \left[\pm \frac{w}{4\varepsilon^2} - \frac{a + w a_w}{4\varepsilon} + \frac{2}{3}(\dot{a} - a^2 u) \pm Z \right] \\ &\approx q_+ q_- \left[\pm \frac{w}{4\varepsilon^2} - \frac{a}{4\varepsilon} + \frac{2}{3}(\dot{a} - a^2 u) \pm \tilde{Z} \right] , \end{aligned} \tag{78}$$

where

$$\tilde{Z} := Z - \frac{a_w (a + w a_w)}{4} .$$

Hence we obtain Ori and Rosenthal's equation (23), viz.,

$$f_{\mathrm{mutual}} \approx -\frac{q_+ q_-}{2\varepsilon} a + \frac{4}{3} q_+ q_- (\dot{a} - a^2 u) \; . \tag{79}$$

They use the term 'bare self-force' to refer to the total EM contribution to the total force f acting on the dumbbell, so not including the external force f_{ext}. We note then that this does not include any divergent terms in the EM self-forces of the two point charges $q_\pm$ which we assume already to have been dealt with by mass renormalisations, as explained above. Hence,

$$\begin{aligned} f_{\mathrm{bare}} &= f_{\mathrm{mutual}} + \hat{f}_+ + \hat{f}_- \\ &= -\frac{q_+ q_-}{2\varepsilon} a + \frac{4}{3} q_+ q_- (\dot{a} - a^2 u) + \hat{f}_+ + \hat{f}_- + O(\varepsilon) \; , \end{aligned} \tag{80}$$

and by (76),

$$f = f_{\mathrm{bare}} + f_{\mathrm{ext}} \; . \tag{81}$$

A.4 Renormalised Equation of Motion

The crucial part of all this comes now, where we require a physical interpretation of f. We begin by noting that the divergent term in f_{bare} in the limit $\varepsilon \to 0$ can be written

$$-\frac{q_+ q_-}{2\varepsilon} a = -E_{\mathrm{es}} a \; , \tag{82}$$

where E_{es} is the electrostatic energy of the dumbbell at rest, viz.,

$$E_{\mathrm{es}} := \frac{q_+ q_-}{2\varepsilon} \; , \tag{83}$$

by which we mean the energy required to bring the two point charges $q_\pm$ together from infinity.

In the authors' own words, the expression for the self-force is to be used to predict the dumbbell motion through an equation of motion of the form

$$f = m_{\mathrm{bare}} a \; , \tag{84}$$

where f is given by (80) and (81). But how do we know from the above that f will have this form, and what is the bare mass? We defined the total force f to be the proper time derivative of the non-EM energy–momentum p of the dumbbell. But in an ICIF such as we know to exist for rigid motion, (70) reads

$$p^\mu := \int_{t=\mathrm{const.}} T^{\mu 0}_{\mathrm{dumb}} \mathrm{d}^3 x^i \; , \tag{85}$$

where $\mathrm{d}^3 x^i$ is the Euclidean volume measure in the hyperplane of simultaneity. The minus sign in (70) goes because the covariant four-velocity is $(-1\,0\,0\,0)$ in the rest frame. Under the rigidity assumption, each component of the matter making up the dumbbell is momentarily at rest in the ICIF, so $T^{i0}_{\mathrm{dumb}} = 0$, $i = 1,2,3$, and $p^i = 0$, $i = 1,2,3$. The dumbbell energy in the rest frame is

$$p^0 := \int_{t=\mathrm{const.}} T^{00}_{\mathrm{dumb}} \mathrm{d}^3 x^i \; , \tag{86}$$

and this is what we define to be the bare mass, i.e.,

$$m_{\mathrm{bare}} := \int_{t=\mathrm{const.}} T^{00}_{\mathrm{dumb}} \mathrm{d}^3 x^i \; . \tag{87}$$

Hence, in the ICIF,

$$p = (m_{\rm bare}\, 0\, 0\, 0)\ . \tag{88}$$

Then in an arbitrary inertial frame in which each component of the matter making up the dumbbell has four-velocity u, we have

$$p = m_{\rm bare} u\ . \tag{89}$$

We appeal to rigidity once again. Since the dumbbell motion is rigid, its composition does not change in time, and we assume that this is enough to deduce that the above bare mass (87) is time independent, i.e.,

$$\frac{\mathrm{d}m_{\rm bare}}{\mathrm{d}\tau} = 0\ .$$

But this allows us to deduce from (89) that

$$f = \frac{\mathrm{d}p}{\mathrm{d}\tau} = m_{\rm bare}\frac{\mathrm{d}u}{\mathrm{d}\tau} = m_{\rm bare} a\ ,$$

which is just the desired equation of motion (84).

All this depends heavily on the motion being rigid. This is precisely what was criticised in [7, Chap. 11], in the discussion of Rohrlich's account of the same ploy for obtaining a covariant four-momentum for the EM fields. This kind of 'electron', if we take it as a crude model for the electron, has to have entirely static innards for this work. It disallows any extension to dynamic systems.

By (81), we now have

$$m_{\rm bare} a = f_{\rm bare} + f_{\rm ext}\ , \tag{90}$$

so defining the renormalised mass and self-force by

$$m_{\rm ren} := m_{\rm bare} + E_{\rm es}\ , \qquad f_{\rm ren} := f_{\rm bare} + E_{\rm es} a\ , \tag{91}$$

we obtain the renormalised equation of motion

$$m_{\rm ren} a = f_{\rm ren} + f_{\rm ext}\ . \tag{92}$$

As noted by Ori and Rosenthal, $m_{\rm ren}$ is precisely the total dumbbell rest energy, including that of the electrostatic interaction. It is also considered, on the basis of this very equation, to be the *observed* mass of the system. This explains the well known dictum in relativistic theories that binding energy must be included in the mass of a composite entity, while no explanation is generally proffered.

Dropping the suffix on the renormalised mass and writing instead $f_{\rm self} := f_{\rm ren}$ for the renormalised self-force, we thus have

$$ma = f_{\rm self} + f_{\rm ext}\ , \tag{93}$$

with

$$f_{\rm self} = f_{\rm bare} + E_{\rm es} a = \frac{4}{3} q_+ q_- (\dot{a} - a^2 u) + \hat{f}_+ + \hat{f}_- + O(\varepsilon)\ , \tag{94}$$

where we have applied (80). This is where we take the limit $\varepsilon \to 0$, ignoring the infinities that crop up inside the renormalised mass m, so we are left with

$$f_{\rm self} = \frac{4}{3} q_+ q_- (\dot{a} - a^2 u) + \hat{f}_+ + \hat{f}_-\ . \tag{95}$$

Of course, the point about the renormalised self-force is that it doesn't have any divergent term. Furthermore, the term it acquires from the dumbbell structure is close to having a universal form once predicted by Dirac [2]. Once again, note that what Ori and Rosenthal refer to as the self-force is precisely this renormalised self-force, i.e., the radiation reaction term in the total self-force.

Here is how the authors finally arrive at Dirac's formula for the radiation reaction, taken more or less verbatim from their paper [9]. Equation (95) relates three unknowns, viz., $\hat{f}_\pm$ and $f_{\rm self}$. Fortunately, we can relate the first two of these to the last by a very general argument. The radiation reaction is a force that a charge q experiences due to its own field, suggesting that it ought to be proportional to q^2. Now in the limit $\varepsilon \to 0$, the worldtube of the whole dumbbell converges to a single worldline and one may assume that the partial self-forces $\hat{f}_\pm$ will satisfy the relation

$$\hat{f}_\pm = \frac{q_\pm^2}{q^2} f_{\rm self} \ .$$

Then

$$\frac{4}{3} q_+ q_- (\dot{a} - a^2 u) = f_{\rm self} - (\hat{f}_+ + \hat{f}_-) = \left(1 - \frac{q_+^2}{q^2} - \frac{q_-^2}{q^2}\right) f_{\rm self} = \frac{2q_- q_+}{q^2} f_{\rm self} \ ,$$

and finally,

$$f_{\rm self} = \frac{4}{3} q^2 (\dot{a} - a^2 u) \ . \tag{96}$$

This is indeed Dirac's expression for the radiation reaction of any charged object in the point particle limit.

A.5 Extended Object with N Point Charges

Here we follow Ori and Rosenthal's account of an extended object with an arbitrary number of point charges under arbitrary rigid acceleration that is as rotationless as possible. Such motion is formulated in the way described in Sect. 3, but in-depth discussions can be found in [6,8]. Most of this section is taken largely verbatim from [9].

The charges are q_s, $s = 1, \dots, N$. A representative point is chosen arbitrarily inside the object. Its worldline is $z(\tau)$, with τ its own proper time, and it has four-velocity $u := \mathrm{d}z/\mathrm{d}\tau$ and four-acceleration $a := \mathrm{d}u/\mathrm{d}\tau$. The worldline of charge q_s is $z_s(\tau)$, also parameterised by τ, and we express it in the form

$$z_s(\tau) = z(\tau) + \varepsilon_s w_s(\tau) \ , \tag{97}$$

where $w_s(\tau)$ is a unit spacelike vector orthogonal to $u(\tau)$ and $\varepsilon_s \geq 0$ is thus the proper distance of charge q_s from the reference point in the instantaneous rest frame of the latter at time τ. Note that there is no sum over s in (97). Charge q_s has proper time τ_s and hence four-velocity $u_s := \mathrm{d}z_s/\mathrm{d}\tau_s$ and four-acceleration $a_s := \mathrm{d}u_s/\mathrm{d}\tau_s$.

As we saw in Sect. 3, rigid motion can be achieved by insisting that the ε_s are constant in time and that the four-vectors w_s are FW transported along the worldline z, i.e.,

$$\dot{w}_s := \frac{\mathrm{d}w_s}{\mathrm{d}\tau} = u(a \cdot w_s) - a(u \cdot w_s) \ .$$

As explained in Sect. 3, FW transport preserves scalar products of four-vectors, so we have $u \cdot w_s = 0$ for all τ, whence

$$\dot{w}_s := \frac{\mathrm{d}w_s}{\mathrm{d}\tau} = u(a \cdot w_s) \ .$$

By the exact same arguments as in Sect. A.1, we conclude as in (61) on p. 48 that

$$\frac{\mathrm{d}\tau_s}{\mathrm{d}\tau} = 1 + \varepsilon_s a \cdot w_s \ . \tag{98}$$

Furthermore, we get the key relations

$$u_s = u \ . \tag{99}$$

These ensure that there is a common ICIF for all the charges and the reference point, a crucial feature of rigid motion. Then, as in (63) on p. 48, we find that the accelerations of all the charges are determined by that of the reference point according to

$$a_s = \frac{a}{1+\varepsilon_s(a\cdot w_s)} \ . \tag{100}$$

The combination of ε_s and w_s without summation over s looks a bit awkward. One is length, the other direction, in a Minkowski sense.

One could also formulate this by selecting at the outset an orthonormal spacelike triad $\{n_i(\tau)\}_{i=1,2,3}$ orthogonal to the worldline of the reference point, then expressing the charge worldlines by

$$z_s(\tau) = z(\tau) + \xi_{si} n_i(\tau) \ . \tag{101}$$

Then we can also write

$$w_s(\tau) = A_{si} n_i(\tau) \ .$$

Then

$$z_s = z + \varepsilon_s w_s = z + \varepsilon_s A_{si} n_i \ ,$$

so

$$\xi_{si} = \varepsilon_s A_{si} \qquad \text{(without summation over } s) \ .$$

We can think of $\mathbf{A}_s$ as a three-vector with components A_{si}, $i=1,2,3$. Likewise for $\boldsymbol{\xi}_s$. Then

$$\boldsymbol{\xi}_s = \varepsilon_s \mathbf{A}_s \qquad \text{(without summation over } s) \ .$$

Each w_s is unit, whence for any fixed s,

$$1 = A_{si} n_i^\mu A_{sj} n_{j\mu} = A_{si} A_{sj} \delta_{ij} = A_{si} A_{si} = \mathbf{A}_s \boldsymbol{\cdot} \mathbf{A}_s \ .$$

So obviously, each $\mathbf{A}_s$ is a unit three-vector. Each $\mathbf{A}_s$ effectively specifies w_s as a linear combination of the spacelike triad and it is important to note that it is constant in time. Of course, since the triad is FW transported, if $\mathbf{A}_s$ is constant, then the corresponding w_s is FW transported. But we have a converse. Since the triad is FW transported, then if $w_s = A_{si} n_i$ is FW transported, we can show that the A_{si}, $i=1,2,3$, are constant in time. This works as follows. Because w_s is FW transported, we have

$$\begin{aligned} \dot w_s &= u(a\cdot w_s) - a(u\cdot w_s) \\ &= A_{si}\Big[u(a\cdot n_i) - a(u\cdot n_i)\Big] \\ &= A_{si}\dot n_s \ , \end{aligned}$$

using the fact that n_s is FW transported. But we also have

$$\dot w_s = \dot A_{si} n_i + A_{si} \dot n_i \ ,$$

whence $\dot A_{si} n_i = 0$. But the n_i are linearly independent.

That was an aside, but we shall use it later. Returning to the main theme here, Ori and Rosenthal are interested in the point particle limit, with the size going to zero but the shape remaining the same. They thus introduce a parameter $\varepsilon > 0$ which denotes the size, e.g., half the maximal distance between pairs of particles in the object. We then define α_s by

$$\varepsilon_s = \varepsilon \alpha_s \ , \qquad 0 < \alpha_s \le 1 \ .$$

The idea then is to take the limit $\varepsilon \to 0$ while keeping all the α_s fixed.

They set out to exploit the results of the dumbbell case, considering all pairs of charges within the system. So for each pair indexed by (s,t), they formulate the central point between them with worldline $z_{st}(\tau)$ given by

$$z_{st}(\tau) = \frac{1}{2}\Big[z_s(\tau) + z_t(\tau)\Big] = z(\tau) + \frac{1}{2}\big(\varepsilon_s w_s + \varepsilon_t w_t\big) \ .$$

where τ is still the proper time of the reference point. Then there exists a positive number $\varepsilon_{st} \geq 0$ and a unit spacelike four-vector $w_{st}(\tau)$ such that

$$\varepsilon_{st} w_{st} = \frac{1}{2}\big(\varepsilon_s w_s + \varepsilon_t w_t\big) \ ,$$

with no summation over s or t. Obviously, $w_{st}(\tau)$ is orthogonal to $u(\tau)$ for every τ, because w_s and w_t are. We can say that ε_{st} is the distance of the central point between q_s and q_t from the reference point in the object, all gauged in the ICIF of that reference point. We allow the possibility $\varepsilon_{st} = 0$, because it could happen that the reference point lies midway between two of the charges. It is also obvious that w_{st} is FW transported along the worldline of the reference point, i.e.,

$$\dot{w}_{st} = (a \cdot w_{st}) u \ ,$$

because it is a linear combination of four-vectors that are also so transported, and ε_s and ε_t are time constant.

The four-velocity and four-acceleration of these central points are

$$u_{st} := \frac{\mathrm{d}z_{st}}{\mathrm{d}\tau_{st}} \ , \qquad a_{st} := \frac{\mathrm{d}u_{st}}{\mathrm{d}\tau_{st}} \ .$$

Now

$$\begin{aligned} u_{st} &= \left(\frac{\mathrm{d}z}{\mathrm{d}\tau} + \varepsilon_{st}\frac{\mathrm{d}w_{st}}{\mathrm{d}\tau}\right)\frac{\mathrm{d}\tau}{\mathrm{d}\tau_{st}} \\ &= \Big[u + \varepsilon_{st}(a \cdot w_{st})u\Big]\frac{\mathrm{d}\tau}{\mathrm{d}\tau_{st}} \\ &= \Big[1 + \varepsilon_{st}(a \cdot w_{st})\Big]u\frac{\mathrm{d}\tau}{\mathrm{d}\tau_{st}} \ , \end{aligned}$$

and we soon obtain the correlates of (98), (99), and (100), viz.,

$$\frac{\mathrm{d}\tau_{st}}{\mathrm{d}\tau} = 1 + \varepsilon_{st}(a \cdot w_{st}) \ , \qquad u_{st} = u \ , \tag{102}$$

and

$$a_{st} = \frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\tau_{st}} = \frac{a}{1 + \varepsilon_{st}(a \cdot w_{st})} \ . \tag{103}$$

Put another way,

$$a_{st}\frac{\mathrm{d}\tau_{st}}{\mathrm{d}\tau} = a \ , \tag{104}$$

which is used later on.

The key point here is that, for each pair (s,t), the three worldlines z_s, z_t, and z_{st} satisfy the same kinematic relations as z_+, z_-, and z for the dumbbell, so that we may apply all the previous results to any such pair. But beware! In the present analysis, z is the worldline of the reference point, while z for the dumbbell corresponds to z_{st}. The dumbbell length $\hat{\varepsilon} = 2\varepsilon$ is now replaced by the distance $\hat{\varepsilon}_{st}$ between q_s and q_t in the ICIF of the reference point.

The four-momentum of the extended object is defined as before [see (70) on p. 51] by

$$p^\mu := -\int_\sigma T^{\mu\nu}_{\mathrm{obj}} u_\nu \mathrm{d}^3\sigma \ . \tag{105}$$

Once again, this is the integral over a hypersurface of simultaneity in the ICIF of the reference point and it does not take into account EM effects. Hence, from one such hypersurface of simultaneity to the 'next', infinitesimally speaking, we expect conservation of energy–momentum to take the form

$$\mathrm{d}p^\mu = \sum_{s=1}^{N} \mathrm{d}p^\mu_s + \mathrm{d}p^\mu_{\mathrm{ext}} + \mathrm{d}p^\mu_{\mathrm{PS}} \ , \tag{106}$$

where $\mathrm{d}p_s$ is the contribution from all the EM forces sourced by *all* the charges in the object acting on q_s, while $\mathrm{d}p_{\mathrm{ext}}$ is the contribution from the total external force, and $\mathrm{d}p_{\mathrm{PS}}$ is the contribution, if any, from Poincaré stresses.

All the comments following (70) on p. 51 remain relevant. This brings us again to the question of the PS self-force. Let us show that $\mathrm{d}p_{\mathrm{PS}} = 0$ when we use the pressure model for the Poincaré stresses and the Ori–Rosenthal way of summing forces acting at different spacetime events. We basically imitate the argument following (71) on p. 52. The only thing is that we now solve the 3D version of the relativistic Navier–Stokes equation in the ICIF, viz.,

$$\mathbf{a} = -\boldsymbol{\nabla} \ln p \ ,$$

assuming once again that the imaginary fluid exerting this 'pressure' between the charges is virtually massless. The incredible thing is that we can do this, thanks to the rigidity assumption.

However, we should immediately note a limitation in the way we must picture this model of the Poincaré stresses. Clearly, we cannot imagine our charges to be immersed in the fluid, for the pressure on them would be the same all round! We have to imagine columns of fluid from each charge to every other charge, explicitly placed there to exercise the necessary attraction or repulsion. This is of course a rather awkward picture, but the fact that it works so well suggests that there ought to be a neater one referring only to stress–energy tensors. Is this where Steane's hidden momentum idea comes in [10]? We return to this in Appendix C.

Here it is conceptually useful to go back to the alternative way (101) of expressing the charge positions, but also imagining a continuum of charges. (This will also be helpful for the last part of [9], which deals with this very situation.) So we had

$$z_s(\tau) = z(\tau) + \xi_{si} n_i(\tau) \ ,$$

where we think of $\boldsymbol{\xi}_s$ as a three-vector in the hyperplane of simultaneity for the reference point at the proper time τ of that reference point. Let us fix τ now, i.e., drop it completely from the notation, and work entirely in the given hyperplane of simultaneity. Then for any $\boldsymbol{\xi}$, we can consider

$$z(\boldsymbol{\xi}) = z(0) + \xi_i n_i \ ,$$

where $z(0)$ is the four-vector denoting the reference point.

Now we can either adapt (21) on p. 13 or (103) obtained a moment ago to deduce that

$$a(\boldsymbol{\xi}) = \frac{a(0)}{1 + \big[a(0)\cdot n_i\big]\xi_i} \ ,$$

where $a(0)$ is the four-acceleration of the reference point. This is indeed a standard result [6,8]. The Navier–Stokes equation thus becomes

$$-\boldsymbol{\nabla} \ln p = \frac{\mathbf{a}}{1 + \big[a(0)\cdot n_i\big]\xi_i} \ ,$$

where $\boldsymbol{\nabla}$ is of course the gradient operator in the $\boldsymbol{\xi}$ space. But $a(0)\cdot n_i$ is just the i th component of $\mathbf{a}$ in the basis given by the triad, which is the one we use to obtain the components ξ_i, so

$$\big[a(0)\cdot n_i\big]\xi_i = \mathbf{a}\cdot\boldsymbol{\xi}\ ,$$

and we thus have

$$\frac{\mathbf{a}}{1+\big[a(0)\cdot n_i\big]\xi_i} = \boldsymbol{\nabla}\ln(1+\mathbf{a}\cdot\boldsymbol{\xi})\ .$$

We have to solve

$$\boldsymbol{\nabla}\ln(1+\mathbf{a}\cdot\boldsymbol{\xi}) = -\boldsymbol{\nabla}\ln p\ ,$$

and the solution is

$$p(\boldsymbol{\xi}) = \frac{k}{1+\mathbf{a}\cdot\boldsymbol{\xi}}\ ,$$

where k is a constant. But of course, in the Ori–Rosenthal way of adding forces acting at different spacetime events, the pressure at $\boldsymbol{\xi}$ is associated with a time dilation factor

$$\frac{\mathrm{d}\tau(\boldsymbol{\xi})}{\mathrm{d}\tau} = 1+\mathbf{a}\cdot\boldsymbol{\xi}\ , \tag{107}$$

where $\tau(\boldsymbol{\xi})$ is the proper time of the point that is always (for all τ) specified by the same $\boldsymbol{\xi}$. Equation (107) is just a variant of relations like (22) on p. 14 or the more recent relation (98). So the effective pressure for this way of summing is just constant and can contribute no self-force between any pair of charges moving rigidly with this object.

This allows us to move from (106) to Ori and Rosenthal's relation (40) on p. 22, viz.,

$$\mathrm{d}p^\mu = \sum_{s=1}^{N}\mathrm{d}p_s^\mu + \mathrm{d}p_{\mathrm{ext}}^\mu\ . \tag{108}$$

The EM energy exchange with q_s is

$$\mathrm{d}p_s = f_s^{\mathrm{EM}}\mathrm{d}\tau_s\ ,$$

where f_s^{EM} is the total EM force acting on that charge, given by

$$f_s^{\mathrm{EM}} = \hat{f}_s + \sum_{t=1,t\neq s}^{N} f_{t\to s}\ ,$$

with $f_{t\to s}$ the EM force that the charge q_t exerts on the charge q_s and $\hat{f}_s$ the partial self-force acting on this charge, i.e., the radiation reaction of q_s on itself.

So we now have

$$\mathrm{d}p = \mathrm{d}p_{\mathrm{ext}} + \sum_{s=1}^{N}\hat{f}_s\mathrm{d}\tau_s + \sum_{s=1}^{N}\sum_{t=1,t\neq s}^{N} f_{t\to s}\mathrm{d}\tau_s\ .$$

The discussion after (76) on p. 53 is relevant here, exploiting (98) on p. 57. With some of the terms, we can replace $\mathrm{d}\tau_s$ by $\mathrm{d}\tau$ with impunity, since we only keep terms up to order ε^0. So we have $\mathrm{d}p_{\mathrm{ext}} \approx f_{\mathrm{ext}}\mathrm{d}\tau$ and we can take f_{ext} to be the usual sum of all contributions to the external force taken simultaneously in the hyperplane of simultaneity of the ICIF, wherever they may act on the system. Likewise, we can replace $\hat{f}_s\mathrm{d}\tau_s$ by $\hat{f}_s\mathrm{d}\tau$ with impunity.

So we now have

$$\mathrm{d}p \approx \left(f_{\mathrm{ext}} + \sum_{s=1}^{N}\hat{f}_s + \sum_{s=1}^{N}\sum_{t=1,t\neq s}^{N} f_{t\to s}\frac{\mathrm{d}\tau_s}{\mathrm{d}\tau}\right)\mathrm{d}\tau\ ,$$

where, as before, the approximation sign means that it is accurate to order ε^0. We take a crucial step here, proposing that the left-hand side should be written

$$\mathrm{d}p = f\mathrm{d}\tau \ ,$$

where $f = m_{\mathrm{bare}}a$ to order ε^0, with a the four-acceleration of the reference point. This is justified as in Sect. A.4 [see (84) on p. 55 and the ensuing discussion]. Naturally, this really is the crucial step, depending heavily on the rigidity of the motion. We conclude that the total force acting on the object is

$$f = \sum_{s=1}^{N} \sum_{t=1, t\neq s}^{N} \frac{\mathrm{d}\tau_s}{\mathrm{d}\tau} f_{t\to s} + \sum_{s=1}^{N} \hat{f}_s + f_{\mathrm{ext}} \ . \tag{109}$$

We refer to the double sum term as the total mutual force

$$f_{\mathrm{mutual}} := \sum_{s=1}^{N} \sum_{t=1, t\neq s}^{N} \frac{\mathrm{d}\tau_s}{\mathrm{d}\tau} f_{t\to s} = \frac{1}{2} \sum_{t\neq s} \frac{\mathrm{d}\tau_{st}}{\mathrm{d}\tau} \left(\frac{\mathrm{d}\tau_s}{\mathrm{d}\tau_{st}} f_{t\to s} + \frac{\mathrm{d}\tau_t}{\mathrm{d}\tau_{st}} f_{s\to t} \right) \ , \tag{110}$$

where the sum on the right-hand side is over all pairs (s, t) with $s \neq t$. Note that $\tau_{st} = \tau_{ts}$.

We now focus on the term in round brackets for a particular pair (s, t). This is where we can bring in the work on the dumbbell. The kinematic relations between the worldlines of the three points z_s, z_t, and z_{st} are exactly the same as those satisfied by the points z_+, z_-, and z describing the dumbbell. This means that (77) and (79) on p. 54 ff imply

$$\frac{\mathrm{d}\tau_s}{\mathrm{d}\tau_{st}} f_{t\to s} + \frac{\mathrm{d}\tau_t}{\mathrm{d}\tau_{st}} f_{s\to t} \approx -\frac{q_s q_t}{\hat{\varepsilon}_{st}} a_{st} + \frac{4}{3} q_s q_t (\dot{a}_{st} - a_{st}^2 u) \ , \tag{111}$$

where $\hat{\varepsilon}_{st}$ is the distance between q_s and q_t (not to be confused with ε_{st}, which is the distance between the reference point in the object and the midpoint between the two charges) and we have used the fact that $u_{st} = u$.

Bear in mind also that the approximation sign in (111) indicates accuracy up to order $\hat{\varepsilon}_{st}^0$, and $\hat{\varepsilon}_{st}$ scales like ε, the measure of the size of the object, so the last relation is accurate up to order ε^0. Now the left-hand side of (111) is multiplied by $\mathrm{d}\tau_{st}/\mathrm{d}\tau$ which we know to be $1 + \varepsilon_{st}(a \cdot w_{st})$ from (102), and when it multiplies the last term in (111), we can therefore replace it by unity with impunity, keeping the same order of accuracy. Likewise, (103) tells us that a_{st} and $\dot{a}_{st}$ can both be replaced by unity in this term.

This is not possible in the first term of (111) because it goes as ε^{-1}. So for the moment we have

$$\frac{\mathrm{d}\tau_{st}}{\mathrm{d}\tau} \left(\frac{\mathrm{d}\tau_s}{\mathrm{d}\tau_{st}} f_{t\to s} + \frac{\mathrm{d}\tau_t}{\mathrm{d}\tau_{st}} f_{s\to t} \right) \approx -\frac{q_s q_t}{\hat{\varepsilon}_{st}} a_{st} \frac{\mathrm{d}\tau_{st}}{\mathrm{d}\tau} + \frac{4}{3} q_s q_t (\dot{a} - a^2 u) \ . \tag{112}$$

But by (104), we know that

$$a_{st} \frac{\mathrm{d}\tau_{st}}{\mathrm{d}\tau} = a \ ,$$

whence

$$\frac{\mathrm{d}\tau_{st}}{\mathrm{d}\tau} \left(\frac{\mathrm{d}\tau_s}{\mathrm{d}\tau_{st}} f_{t\to s} + \frac{\mathrm{d}\tau_t}{\mathrm{d}\tau_{st}} f_{s\to t} \right) \approx -\frac{q_s q_t}{\hat{\varepsilon}_{st}} a + \frac{4}{3} q_s q_t (\dot{a} - a^2 u) \ . \tag{113}$$

All the kinematic quantities in this last relation are those associated with the representative point in the object. The only reference to the two charges in question is through q_s, q_t, and $\hat{\varepsilon}_{st}$.

Equation (110) now reads

$$f_{\rm mutual} \approx -E_{\rm es}a + \frac{2}{3}\sum_{s\neq t} q_s q_t(\dot{a} - a^2 u)\ ,$$

where

$$E_{\rm es} := \frac{1}{2}\sum_{s\neq t}\frac{q_s q_t}{\hat{\varepsilon}_{st}}$$

is precisely the electrostatic energy of the system of N charges. The total bare self-force is the first two terms in (109), viz.,

$$\begin{aligned} f_{\rm bare} &= f_{\rm mutual} + \sum_{s=1}^{N}\hat{f}_s \\ &= -E_{\rm es}a + \frac{2}{3}\sum_{s\neq t} q_s q_t(\dot{a} - a^2 u) + \sum_{s=1}^{N}\hat{f}_s + O(\varepsilon)\ . \end{aligned}$$

We now go back to (109), which tells us that

$$f = f_{\rm bare} + f_{\rm ext} = -E_{\rm es}a + \frac{2}{3}\sum_{s\neq t} q_s q_t(\dot{a} - a^2 u) + \sum_{s=1}^{N}\hat{f}_s + f_{\rm ext} + O(\varepsilon)\ ,$$

then define

$$m_{\rm ren} = m_{\rm bare} + E_{\rm es}\ , \qquad f_{\rm ren} = f_{\rm bare} + E_{\rm es}a\ ,$$

and take the limit $\varepsilon \to 0$ to obtain

$$m_{\rm ren}a = f_{\rm ren} + f_{\rm ext} = \frac{2}{3}\sum_{s\neq t} q_s q_t(\dot{a} - a^2 u) + \sum_{s=1}^{N}\hat{f}_s + f_{\rm ext}\ . \tag{114}$$

As before, Ori and Rosenthal also refer to the renormalised self-force $f_{\rm ren}$ simply as the self-force $f_{\rm self}$ because, for them, the EM self-force refers only to the radiation reaction term in the EM self-force, the only term that survives renormalisation and the point particle limit.

From the point of view of understanding inertial mass, it is the left-hand side of (114) that interests us, before we take the limit $\varepsilon \to 0$, since we can understand how EM effects contribute to inertial mass when objects are *not* point particles. What we see is that we have the standard form of a relativistic equation of motion, just like the relativistic form of Newton's second law, in which the parameter corresponding to the mass is in fact $m_{\rm bare}+E_{\rm es}$. We thus understand the old relativistic adage that binding energies must be included in the inertial masses of bound state particles.

What interests Ori and Rosenthal is to obtain the radiation reaction $f_{\rm self}$ given by the first two terms in (114). Once again, they exploit the idea that this is universal and should be quadratic in the charge, which prompts the ansatz

$$\hat{f}_s = \frac{q_s^2}{q^2} f_{\rm self}\ ,$$

where $q := \sum_s q_s$ is the total charge. This implies that

$$\left(q^2 - \sum_{s=1}^{N} q_s^2\right) f_{\rm self} = \frac{2}{3} q^2 \sum_{s\neq t} q_s q_t(\dot{a} - a^2 u)\ ,$$

at which point we note that

$$q^2 - \sum_{s=1}^{N} q_s^2 = \sum_{s \neq t} q_s q_t \ ,$$

and hence, finally,

$$f_{\text{self}} = \frac{2}{3} q^2 (\dot{a} - a^2 u) \ ,$$

the standard universal formula for the radiation reaction as obtained by Dirac. The equation of motion is thus

$$ma = \frac{2}{3} q^2 (\dot{a} - a^2 u) + f_{\text{ext}} \ ,$$

where we now denote the renormalised mass simply by m.

A.6 Continuous Charge Distribution

The extension to this case is more or less immediate provided that we can somehow adapt the 'pressure' model to show that there will be no Poincaré stress self-force. The picture we have to adopt is at least as awkward as in the case of a finite number of charges, but probably not more so, since we proceed by dividing the charge distribution up into a finite number of charge elements and citing the result already proven. Clearly, however, it is not a very physical argument. On the other hand, it strongly suggests that there should be some more elegant route to the same result passing directly by suitable stress–energy tensors.

A.7 Summary and Conclusion

Ori and Rosenthal [9] have shown that one may use an alternative definition of the EM self-force to obtain an equation of motion for the charge dumbbell in the point particle limit that is entirely consistent with the usual definition. Here we point out with Steane [10] that the self-force due to internal forces (Poincaré stresses) must be zero when we use this same altered definition for sums of forces acting at different spacetime events, and we show by generalising a method due to Steane [10] that this will indeed be the case, provided that the motion of the dumbbell is rigid. We also suggest that this whole method depends heavily on the rigidity of the motion and would not be generalisable to 'particles' whose internal structure involves a dynamic equilibrium, such as the proton.

We argue against Steane's proposal [10] that Ori and Rosenthal's method is merely a natural choice for a certain kind of observer, namely one comoving with the dumbbell. What matters here is the equation of motion of the system and its resulting worldtube. In the present case, when the various assumptions are satisfied and the approximations justified, we obtain exactly the same worldline in the point particle limit as we would with the previous method, so this clearly constitutes a consistent reanalysis of the problem that has nothing to do with any observer. Outside the point particle limit, there are likely to be some small differences between the predictions due to the different approximations.

We note that Ori and Rosenthal's method remains valid for an N particle object in rigid acceleration and also for a continuous distribution of charge, since we can still provide a plausible argument to show that the internal Poincaré stresses will not contribute their own self-force when we consistently use Ori and Rosenthal's altered definition for sums of forces acting at different spacetime events. This argument exploits the rigidity of the motion to deduce a 'pressure' model for these stresses that is totally determined by the kinematics of rigid motion.

B Extending the Usual Definition of the EM Self-Force to Systems with N Charges and Continuous Charge Distributions

It is clear that, since we managed to get the pressure model to work for the charge dumbbell, we ought to be able to get it to work for any finite system of point charges in rigid motion, simply by following a similar formulation to Ori and Rosenthal, but using Steane's first method for summing forces acting at different spacetime events (see Sect. 9.1) [10].

Here we focus solely on the mass renormalisation term in the self-force. In the calculation of Sect. 4, we referred everything to the worldline of one of the dumbbell charges. With N charges present, we need to choose a reference point in the object. In fact, we can follow the notation of Ori and Rosenthal here, labelling the charges q_s for $s = 1, \ldots, N$, and writing down an equation of motion for each charge in the form

$$m_s a_s = f_s^{\text{ext}} + \sum_{t=1, t\neq s}^{N} f_{t\to s}^{\text{EM}} + \sum_{t=1, t\neq s}^{N} f_{t\to s}^{\text{PS}} + f_s^{\text{RR}} , \tag{115}$$

where m_s is the mass of q_s, a_s is its acceleration, f_s^{RR} is the radiation reaction of q_s on itself, $f_{t\to s}^{\text{EM}}$ is the EM force of q_t on q_s, and $f_{t\to s}^{\text{PS}}$ is the Poincaré stress of t on s. It should be noted that the m_s are not bare. They already take into account renormalisation due to their own self-force.

How are we to understand $f_{t\to s}^{\text{PS}}$? We are once again imagining columns of massless fluid that somehow manage to maintain the equilibrium, as required by the constraint of rigid motion, between pairs of charges q_s and q_t. So having selected one charge q_s, we consider that every other charge q_t will exert such a Poincaré stress on q_s via such a column of fluid.

Note that, although a simple pressure in the fluid would be enough to keep two unlike charges apart, the self-force effect here is *not* simply due to a pressure force along the line joining q_t to q_s. Rather it is due to a pressure *difference* between the locations of q_s and q_t which would lead to a *net pressure self-force* on the combined system of the two charges. And this pressure difference along the imaginary column of fluid arises because of the constraint of rigid motion.

The N equations (115) are analogous to (40) on p. 23. Since

$$a_s = \frac{a}{1 + \xi_{si}(a \cdot n_i)} = \frac{a}{1 + \boldsymbol{\xi}_s \cdot \mathbf{a}} ,$$

using the notation developed in Sect. A.5 [see (100) on p. 58 and thereafter], i.e.,

$$\boldsymbol{\xi}_s := (\xi_{s1}, \xi_{s2}, \xi_{s3}) , \qquad \mathbf{a} := (a \cdot n_1, a \cdot n_2, a \cdot n_3) ,$$

and since the ξ_{si} are $O(\varepsilon)$, we have

$$a_s = a + O(\varepsilon) .$$

So keeping only terms to order ε^0, one thing we can do is to sum all the equations in (115) to obtain

$$m_{\text{bare}} a \approx \sum_s f_s^{\text{ext}} + \sum_{t\neq s} f_{t\to s}^{\text{EM}} + \sum_{t\neq s} f_{t\to s}^{\text{PS}} + \sum_s f_s^{\text{RR}} , \tag{116}$$

where $\sum_{t\neq s}$ is the sum over all pairs (t, s) with $t \neq s$, the approximation sign means there may be other terms $O(\varepsilon)$, and we have defined

$$m_{\text{bare}} := \sum_s m_s ,$$

the bare mass of the whole object. Note that it is bare in the sense that we have not yet implemented renormalisation to account for the structure of the present N particle object.

Equation (116) is the counterpart of (106) on p. 60. The first term on the right-hand side is the total external force, where the sum is of course taken here using Steane's first method for summing forces acting at different spacetime events (see Sect. 9.1), the one we used throughout the first part of this paper, which is just the straight sum of the forces evaluated in an instantaneously comoving inertial frame (ICIF) for the whole system. This is indeed how we intend the sum in (116). We write

$$f_{\text{ext}} := \sum_s f_s^{\text{ext}} \ .$$

The interesting thing is of course the sum over pairs (s, t) for $s \neq t$, which can be written

$$\frac{1}{2} \sum_{t \neq s} \left(f_{t \to s}^{\text{EM}} + f_{s \to t}^{\text{EM}} + f_{t \to s}^{\text{PS}} + f_{s \to t}^{\text{PS}} \right) . \tag{117}$$

But each such pair is of course a charge dumbbell moving rigidly, so we can directly apply the results of Sect. 7. Rigidity is doing a lot of work here. Not only does it give us the rigidly moving dumbbells, each in its own ICIF, but it gives us all the dumbbells moving rigidly together in a common ICIF for the whole system.

Basically, all we need from Sect. 7 is (45) on p. 26, which allows us to replace the spatial part of the sum in (117) by

$$-\frac{1}{2} \sum_{s \neq t} \frac{q_s q_t}{d_{st}} \mathbf{a} + O(\varepsilon) = -E_{\text{es}} \mathbf{a} + O(\varepsilon) \ ,$$

where d_{st} is the proper distance from q_s to q_t in the ICIF and

$$E_{\text{es}} := \frac{1}{2} \sum_{s \neq t} \frac{q_s q_t}{d_{st}}$$

is the electrostatic energy of the system of N charges due to their association in the present system. From here, renormalisation of the equation of motion proceeds as usual.

Clearly, in principle, this result extends to arbitrary continuous distributions of charge in arbitrary rigid motion, dividing them up into finite numbers of charge elements and taking a limit as the size of the charge elements goes to zero. The physical picture is awkward, however, given the way we must picture implementation of the pressure model. It would be better to have some more elegant model for the Poincaré stresses, something that might explain why the Navier–Stokes equation delivers such a helpful result here.

C Hidden Momentum Solution to Misaligned EM Self-Force

Steane mentions hidden momentum in his paper [10] and discusses it with examples in [11] (see in particular Chap. 16 of the latter). In fact, whenever a force in one place does work that appears as energy in another, e.g., when we pull something along with a rope, or when we find energy entering a system at a certain rate at one point and being removed from it at another, this flow of energy can be identified with a momentum. Steane gives a careful description of the celebrated Trouton–Noble experiment to illustrate this [11, Chap. 16].

The charge dipole or dumbbell is another case in point. If there is a three-force $\mathbf{f}$ on one end moving with three-velocity $\mathbf{v}$, then energy is entering the system at the rate $\mathbf{f} \cdot \mathbf{v}$. If also there is a three-force $-\mathbf{f}$ on the other end moving with three-velocity $\mathbf{v}$, then energy is being removed from the system at the rate $\mathbf{f} \cdot \mathbf{v}$. We thus view energy as flowing at this rate along the rod joining the two charges. What momentum would this correspond to?

If the rod has length L and energy takes time T to transfer, we have a mass equivalent $\mathbf{f}\cdot\mathbf{v}/c^2$ per unit time flowing down the rod at speed L/T, and that would correspond to a momentum

$$\frac{\mathbf{f}\cdot\mathbf{v}}{c^2}\times\frac{L}{T}\times T\ .$$

This therefore implies a momentum of $\mathbf{f}\cdot\mathbf{v}/c^2$ per unit length of rod. In the case of the charge dumbbell of Sect. 4, we have something that corresponds approximately to this situation, because the three-velocity of each end of the rod is roughly the same, while the main part of the EM force is reversed at each end. Let us see how the calculation works out in the details and then comment, in particular on what this means physically, and also on the prospects for a smoother transition of the calculation to the cases of many point charges and continuous charge.

First of all, we have the four-force

$$F^{\mu}(A\text{ on }B)=e_BF_A^{\mu\nu}(B)v_\nu^B=e_Ae_B\frac{u^\mu}{d^2}+O(d^{-1})\ ,$$

from (23) on p. 14 and (25) on p. 15, where u is the unit spacelike four-vector from A to B. The rate of work done on B by this force is

$$\begin{aligned}\mathbf{F}\cdot\mathbf{v}_B&=-e_BF_A^{i\nu}(B)v_\nu^Bv_i^B/\gamma(v_B)^2\\&=+e_BF_A^{0\nu}(B)v_\nu^Bv_0^B/\gamma(v_B)^2\\&=+\frac{e_Ae_Bu^0}{d^2\gamma(v_B)}+O(d^{-1})\ ,\end{aligned}\tag{118}$$

where we sum over $i=1,2,3,$ in the first line and we use the antisymmetry of the Faraday tensor to get the second line. The symbol $\gamma(v_B)$ is of course just the usual gamma factor, hence equal to the zero component v_B^0 of the four-velocity v_B. The symbols $\mathbf{F}$ and $\mathbf{v}_B$ denote the three-force on B and three-velocity at B, respectively.

We now divide by $c^2=1$ to get the corresponding three-momentum per unit length in the rod, and multiply by the length

$$\gamma(v_B)^{-1}d+O(d^2)$$

to get the total hidden momentum. It is because this is order d that we only require the rate of energy transfer to order d^{-2}. The hidden momentum is thus

$$\mathrm{HM}=\frac{e_Ae_Bu^0}{\gamma(v_B)^2d}+O(d^0)\ .\tag{119}$$

What about the $O(d^0)$ contribution to this? We do not need it for the present calculation because we are only trying to explain the $O(d^{-1})$ term in the total self-force. To compare the hidden momentum with that force, we require its rate of change with respect to time in the chosen inertial frame, viz.,

$$\frac{\mathrm{d(HM)}}{\mathrm{d}t}=\frac{e_Ae_B}{d}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{u^0}{\gamma(v_A)^2}\right]+O(d^0)\ ,$$

where we replace $\gamma(v_B)$ by $\gamma(v_A)$, valid to this order in d. This would be directed along the rod, hence parallel to the three-vector $\mathbf{u}$.

We would like to show that this is equal to the spatial component of the three-force corresponding to the unwanted part of the EM self-force (27) on p. 15, viz.,

$$\frac{e_Ae_B}{d\gamma(v_A)}(u\cdot\dot{v}_A)\mathbf{u}+O(d^0)\ ,$$

so let us examine $u \cdot \dot{v}_A$. We express the four-vectors in terms of the tetrad of Sect. 3, so

$$u := \frac{x_B - x_A}{d} = \frac{\xi^i n_i}{d} ,$$

where ξ^i, $i = 1, 2, 3$, are fixed numbers. Hence,

$$u^0 = \frac{\xi^i}{d} n_i^0 .$$

Furthermore, by (11) on p. 10,

$$n_i(\tau_A) \cdot \dot{v}_A(\tau_A) = -a_{0i}(\tau_A) ,$$

so

$$u \cdot \dot{v}_A = \frac{\xi^i}{d} n_i \cdot \dot{v}_A = -\frac{\xi^i a_{0i}}{d} ,$$

and by (16) on p. 12, the proper time derivative of the space triad is

$$\dot{n}_i^\mu = a_{0i} v_A^\mu .$$

Hence,

$$\frac{\mathrm{d}u^0}{\mathrm{d}t} = \frac{\mathrm{d}u^0}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\xi^i}{d}\dot{n}_i^0 \frac{1}{\gamma(v_A)} = \frac{\xi^i a_{0i} v_A^0}{\gamma(v_A) d} = \frac{\xi^i}{d} a_{0i} .$$

So the rate of change of hidden momentum, which we know to lie along $\mathbf{u}$, is

$$\frac{e_A e_B}{\gamma(v_A)^2 d}\frac{\xi^i}{d} a_{0i} + \frac{e_A e_B}{d}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\gamma(v_A)^2}\right] u^0 , \tag{120}$$

while the excess term in the EM self-force is

$$-\frac{e_A e_B}{\gamma(v_A) d}\frac{\xi^i}{d} a_{0i}\mathbf{u} . \tag{121}$$

The first thing to note is that there is no hope of these things being equal in general. However, they would be equal in the instantaneous rest frame, provided that we understand the signs. The point is that $\mathbf{u}$ is then a unit three-vector and $u^0 = 0$, while we also have $\gamma(V_A) = 1$. But we have to understand the signs here, because physically we wish to say that this excess term is driving the change in the hidden momentum associated with internal pressure forces. We need to consider a specific situation to see how this works.

For example, if e_A and e_B have the same sign so that each pushes the other away, and if the angle between $\boldsymbol{\xi}$ and $\mathbf{a}_0$ is acute, so that the acceleration has a component in the direction from A to B, then the excess term in the self-force points from B to A. Under the same conditions, there are still two cases, depending on whether the three-velocity of B has a component from A to B or from B to A. In the former case, positive work is done at B, so that energy goes into the rod there and is removed at A, whence the hidden momentum is conceived as going from B to A. In the latter case, it is conceived as going from A to B. In each case, we need to assess the direction of the rate of change of this hidden momentum to check that it really is always from B to A, to match the excess term in the self-force.

When $\mathbf{v}_B$ has a component from A to B, we deduce from (118) that u^0 is positive so, by (119), HM has positive magnitude $e_A e_B u^0/d$ from B to A. Then by (120), this magnitude has rate of change $e_A e_B \boldsymbol{\xi} \cdot \mathbf{a}_0/d^2$ which is positive by the present hypothesis, so the direction of the rate of change of HM is from B to A, as hoped.

When $\mathbf{v}_B$ has a component from B to A, we deduce from (118) that u^0 is negative so, by (119), HM has positive magnitude $-e_A e_B u^0/d$ from A to B. Then by (120), this magnitude has rate of change $-e_A e_B \boldsymbol{\xi} \cdot \mathbf{a}_0 / d^2$ which is negative by the present hypothesis, so the direction of the rate of change of HM is from B to A, as hoped once again.

Of course, there are plenty of other cases to consider, with $\boldsymbol{\xi} \cdot \mathbf{a}_0 < 0$ and $e_A e_B < 0$. And even if we did not have the unpleasant second term in (120) and the extra gamma factor in the other, we would still not be quite through with explaining what the excess term in the self-force is doing, because although we have established that the rate of change of the hidden momentum has magnitude (120) in the right direction along $\mathbf{u}$ to match the excess term (121) in the self-force, and although u is a unit spacelike vector with $u^2 = -1$, as long as we are in a frame with $u^0 \neq 0$, we will not have $\mathbf{u}^2 = 1$. In fact, we will only have $\mathbf{u}^2 = 1$, which we absolutely require to clinch the equality of the rate of change of hidden momentum and the excess term in the self-force, if we choose an inertial frame that is instantaneously at rest relative to the dumbbell.

So it is in such frames that we finally carry out all the above calculation. This makes sense to some extent. After all, we summed the EM forces of A on B and of B on A in the simultaneity plane of an instantaneous rest frame, so we expect to examine hidden momentum as it would be in such a frame. However, the argument remains messy and not particularly convincing, even though some aspects of the expressions point toward a message of some kind here.

One can then imagine finite constructions of point charges in rigid motion and understand hidden momentum as flowing between them at various rates in just such a way as to obtain a total contribution to inertial mass from EM self-forces that is independent of the shape of the construction and independent also of the relation between the direction of acceleration and the shape, and corresponds precisely to the equivalent mass of the EM fields. This then extends also to continuous charge distributions in rigid motion.

Those are big steps and the generality of the calculation in this appendix does nothing to enhance its clarity. An interesting exercise is to turn to the much simpler case of axial linear acceleration discussed in [7, Chap. 7] and see how the hidden momentum argument works there, especially since the four self-force calculations in [7] operate in a quite different way to those in the present paper, or the paper by Ori and Rosenthal. Indeed, in [7], each calculation is carried out in an arbitrary inertial frame and the EM force of A on B is added to the EM force of B on A *at the same time in the chosen inertial frame*. In this picture, the hidden momentum argument seems to fail due to gamma factors that turn up in the wrong place.

Here are the details for those who have access to [7, Chap. 7]. We set $c = 1$ throughout and the two charges are e_A and e_B. The electric three-force of A on B is

$$\mathbf{F}^{\mathrm{elec}}(\mathrm{A\ on\ B}) = e_A e_B \frac{1 + v_+^{\mathrm{A}}}{(r_+^{\mathrm{AB}})^2 (1 - v_+^{\mathrm{A}})} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ,$$

where v_+^A is the speed of A in the positive x direction at the retarded time t_+^A and r_+^{AB} is the spatial separation between B now (at time t) and A at the retarded time t_+^A, which is just $r_+^{AB} = t - t_+^A$. There is no magnetic three-force on B due to the fields of A.

Work is thus done on B at the rate

$$\mathbf{F}^{\mathrm{elec}}(\mathrm{A\ on\ B}) \cdot \mathbf{v}_B = e_A e_B \frac{1 + v_+^{\mathrm{A}}}{(t - t_+^A)^2 (1 - v_+^{\mathrm{A}})} v_B ,$$

where $v_B = v_B(t)$ is the x component of the three-velocity of B now (at time t). Dividing by $c^2 = 1$, we get the hidden three-momentum (HM) per unit length, so we multiply by $\gamma(v)^{-1} d$ to get

$$\mathrm{HM} = e_A e_B \frac{d}{\gamma(v_A)} \frac{1 + v_+^{\mathrm{A}}}{(t - t_+^A)^2 (1 - v_+^{\mathrm{A}})} v_B .$$

Of course, $d/\gamma(v_A)$ is just an estimate of the length in the given inertial frame, but it is good enough to this order in d. By the expansions in [7, Chap. 7],

$$\frac{1}{(t-t_+^A)^2} = \frac{\gamma^2(1-v)^2}{d^2} + O(d^{-1}) \;,$$

$$\frac{1+v_+^{\mathrm{A}}}{1-v_+^{\mathrm{A}}} = \frac{1+v}{1-v} + O(d) \;,$$

and

$$v_B(t) = v(t) - \gamma v a d \;,$$

where $v := v_A(t)$, $a = \dot{v}(t)$, and $\gamma = (1-v^2)^{-1/2}$. Putting that together, we obtain

$$\begin{aligned} \mathrm{HM} &= e_A e_B \frac{d}{\gamma}\left[\frac{\gamma^2(1-v)^2}{d^2} + O(d^{-1})\right]\left[\frac{1+v}{1-v} + O(d)\right] v(1-\gamma a d) \\ &= e_A e_B \frac{1}{\gamma}\frac{\gamma^2(1-v^2)}{d} v + O(d^0) \\ &= \frac{e_A e_B}{\gamma d} v \;. \end{aligned}$$

This is a three-momentum. The three-force required to cause the necessary rate of change of this hidden momentum with respect to time would be its time derivative

$$F^{\mathrm{hidden}} = \frac{e_A e_B}{d}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v}{\gamma}\right) \;.$$

Unfortunately, what we would like to see here is $\mathrm{d}(\gamma v)/\mathrm{d}t$ on the right-hand side. That would be perfect for renormalising the relativistic version of Newton's second law, as explained in [7, Chap. 7].

What can we conclude from that? The calculation of the self-force is very different in [7] because we sum the forces at different spacetime events in an arbitrary inertial frame that is not the rest frame of the system. It would be interesting to know how that should affect hidden momentum arguments. When we sum in the instantaneous rest frame, these arguments do give something plausible, although the details are worryingly messy. On the whole, although it must be giving some kind of insight, this idea does not seem to improve much on the picture given by the explicit pressure model discussed earlier.

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