Self-Force and Inertia. Old Light on New Ideas

Stephen N. Lyle

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The aim here will be to revive an old idea that has been around now for over 100 years, since before the advent of relativity theory. The subjects will be:

- inertial mass,

- Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$,

- the dichotomy between spatially extended and point particles,

and

- the idea of classical mass renormalisation as introduced by Dirac in 1938.

We begin with a simplistic model of the electron as a uniform spherical shell of charge of radius a. When the charge shell is sitting still in an inertial frame, we can calculate the energy in its Coulomb fields, and we obtain the value

$$\text{Energy in EM fields} = \frac{e^2}{2a} \, ,$$

where e is the charge distributed over the shell. Today we would automatically associate a mass with this energy by dividing by c^2 :

$$\text{Mass associated with EM field energy} = \frac{e^2}{2ac^2} \, . \tag{1}$$

We can call this the energy-derived mass of the fields.

Now there is an obvious problem here if we let a tend to zero, since both this energy and the associated mass tend to infinity. And this raises the question as to whether it is ever justified to take the point particle limit for a charged particle.

We can also calculate the three-momentum of the EM fields produced by the charge shell. When it is sitting still in an inertial frame, there is no momentum in the fields, but when it is moving with constant velocity $\boldsymbol{v}$, we find

$$\boldsymbol{p} = \frac{4}{3} \frac{e^2}{2ac^2} \boldsymbol{v} \, .$$

This is the momentum of a particle of mass

$$\text{Momentum-derived mass of EM fields} = \frac{4}{3} \frac{e^2}{2ac^2} \, . \tag{2}$$

Note that the momentum-derived mass of the fields of the charged shell is equal to 4/3 of the energy-derived mass, a point we shall return to later.

If the charge shell is moving very fast, but still at constant velocity, in an inertial frame, we can do a relativistic calculation of the three-momentum in the EM fields and we obtain exactly the same thing as before except that now there is a relativistic factor $\gamma(v)$, which is an increasing function of the speed v :

$$\boldsymbol{p} = \frac{4}{3} \frac{e^2}{2ac^2} \gamma(v)\boldsymbol{v} \, , \quad \gamma(v) := \frac{1}{\sqrt{1 - v^2}} \, .$$

This is the momentum of a particle of mass

$$\text{Momentum-derived mass of EM fields} = \frac{4}{3}\frac{e^2}{2ac^2}\gamma(v) \ .$$

So what we discover here is that the momentum-derived mass of the EM fields due to this charged shell increases with speed v in exactly the way one would expect the inertial mass of a particle to increase with speed in relativistic dynamics.

Note that we have to assume that the charge shell FitzGerald contracts in the direction of motion and this reminds us that there must be some binding forces in this system. Indeed the various elements of negative charge distributed over the spherical surface will tend to repel one another and some other force will be needed to hold these elements in place. The actual shape of the charge distribution will depend on the balance between the repulsive EM forces between the charge elements and these binding forces.

Well so far we have talked about the energy and momentum in the fields of a charged shell and the associated masses, but how do we actually predict the resistance that something will show to being accelerated, as quantified by its inertial mass?

In modern particle physics there are basically two kinds of particle:

- Truly elementary particles like quarks and leptons (the electron is an example of a lepton) which are not considered to be made of smaller particles.

- A whole host of bound state particles.

A good example of the latter would be the proton, a bound state of three quarks, according to modern theory. Now if we had to estimate the mass of a proton, we would certainly want to include the rest masses of the constituent quarks, but we would also want to include the kinetic energy of those quarks, and also any strong, weak, or electromagnetic binding energy involved in the system, with the energies being suitably divided by c^2. Here, of course, we are making ample use of the celebrated relation $E = mc^2$.

But what about the electron? If it really is a point particle, as often assumed, we cannot make a model for its inertia that is intrinsic to its structure in order to predict its inertial mass. So for the truly elementary particles like quarks and leptons, we invent a field called the Higgs field, and we arrange for these particles to interact with that field in such a way that moving through it is rather like moving through honey, according to one analogy. Put another way, the elementary particles get their inertia from the outside.

However, we still need to renormalise the electron mass in quantum electrodynamics (QED), and we need to renormalize the masses of the other elementary particles likewise in the sophisticated quantum field theories appropriate to them. This suggests that we should not treat the electron as a point particle. At least that would save us the trouble we noted for the charge shell model of the electron in the classical case, as discussed above. So for the purposes of this discussion, let us assume that there are in fact no elementary particles, i.e.,

that all particles do in fact have some structure, and make the rather radical bootstrap hypothesis that it is the very structure of each particle that causes it to resist being accelerated.

It should be noted that at the present time we use a hybrid model for bound state particles. For example, the quarks in the proton get their mass by interacting with the Higgs field, but most of the mass of the proton comes from its internal structure. It should be noted in this context that ab initio determinations of the light hadron masses are now possible using lattice quantum chromodynamics (QCD) [1]. The hadrons are strongly interacting particles like the proton and the neutron, and QCD is our best theory for the strong force. Using very sophisticated computer simulations, these calculations do indeed confirm that most of the mass of the proton and other similar bound state particles comes from their internal structure.

With the above bootstrap hypothesis in mind, could it be then that all the mass of the electron comes somehow from the electrodynamic effect, i.e., from the mass associated with its EM fields? If this were to be true, we would have to have

$$m_\mathrm{e} \sim \frac{e^2}{ac^2} \ ,$$

where m_e is the electron mass, e the electron charge, c the speed of light, and a the linear dimension of the charge distribution being used to model the electron. This can be rearranged to estimate the latter:

$$a \sim \frac{e^2}{m_\mathrm{e}c^2} = r_\mathrm{classical} \ ,$$

which is called the classical electron radius. The latter can be calculated from the measured values of the constants in the above expression, whence

$$r_\mathrm{classical} = 2.82 \times 10^{-15} \ \mathrm{m} \ .$$

Unfortunately this is much too big. Experiment suggests that the linear dimension of the electron cannot be greater than 10^{-18} m, so it would have to be at least a thousand times more massive for this to work.

Another problem is the strange discrepancy between the energy-derived and momentum-derived EM masses, i.e., the factor of $3/4$ in the relation

$$m_\mathrm{EM}^\mathrm{EDM} = \frac{3}{4}m_\mathrm{EM}^\mathrm{MDM} \ ,$$

as can be seen from (1) and (2). This has caused a long controversy, still underway in some quarters. However, the basic explanation was pointed out by Poincaré over a hundred years ago! It is due to leaving out the binding forces in the system. The point is that the EM energy–momentum tensor for the charge shell system is not conserved everywhere, and one cannot obtain a Lorentz covariant four-momentum by integrating a non-conserved energy–momentum tensor over spacelike hypersurfaces.

4

One approach here is to redefine the EM energy–momentum tensor, or the energy–momentum of the EM fields, in an ad hoc way so that things work out. But another is to say that the discrepancy *should* be there in general until we include *all* the forces involved in the particle. Then we obtain a total energy–momentum tensor which is conserved, and we can integrate that over spacelike hypersurfaces to obtain an energy–momentum four-vector which is Lorentz covariant.

So whatever happened to electromagnetic mass? Feynman thought that as soon as one had to introduce unknown binding forces into the model for the electron, that made the whole idea too complicated to be worth bothering about. And then of course quantum electrodynamics came along and that deals with the electron in a very different way, although as mentioned earlier, it does leave us with some of the same problems, in particular, the problem of mass renormalization.

And what about a mechanism here? Why should these electromagnetic effects cause a spatially extended charge distribution to resist being accelerated? So far we have talked about the EM fields around the charge shell, found their energy and divided it by c^2 to associate a mass with that. And we have found the momentum of the EM fields, but not the momentum of the charge shell itself. However, we may ask what brings about this momentum.

Remember that, when the charge shell is sitting still in an inertial frame, there is no momentum in the fields. But if we push it for a while and get it moving, we find that there is some momentum in the fields, and this suggests that we must have supplied some force in addition to the one required by the mechanical inertial of the electron, by which we mean any inertia from other origins than the electromagnetic effects. But that in turn suggests that there must have been a corresponding extra force acting back on the accelerating agent.

And this is indeed the case. Whenever we try to accelerate a spatially extended charge distribution, it will exert an EM force on itself, in fact, an EM self-force. Here is the formula Feynman gives in the Feynman Lectures [2]:

$$ F_{\text{self}} = -\alpha \frac{e^2}{ac^2}\ddot{x} + \frac{2}{3}\frac{e^2}{c^3}\dddot{x} + \beta\frac{e^2 a}{c^4}\ddddot{x} + O(a^2) \ . \tag{3} $$

What we have here is the EM self-force for a rather arbitrarily shaped spatially extended charge distribution or charge blob of linear dimension a, moving in an arbitrary way in one dimension. The symbol x denotes the position of some preselected point within the charge blob, a function of the proper time τ along the worldline of that point. Dots over the x denote proper time derivatives, so $\dot{x}$ is the speed, $\ddot{x}$ is the acceleration, and so on.

The self-force has been expanded as a power series in a, with a leading order term that goes as a^{-1}, then a term that is independent of a, then a whole infinite sum of terms going as a, a^2, and so on. The total charge on the charge blob is e, and α and β are constants that depend more or less only on the shape of the charge distribution (although not quite, as we shall see). For example, α has the value 2/3 for the spherical charge shell.

5

The first thing to note is that, if we try to take a point particle limit by letting a tend to zero, that leading order term goes to infinity, so the EM self-force is always infinite in the point particle limit.

The second thing to note is that the leading order term, going as a^{-1}, is proportional to the acceleration $\ddot{x}$. This will be very important when we come to consider mass renormalization in a moment. The coefficient of the acceleration in this leading order term has units of mass and we may call it the self-force-derived mass of the charge blob:

$$m_{\text{EM}}^{\text{SFDM}} = \alpha \frac{e^2}{ac^2} \ . \tag{4}$$

This can be compared with the energy-derived and momentum-derived masses in (1) and (2).

We can think of the self-force as expressing a breakdown of Newton's third law within the particle, in the sense that the sum of all the electromagnetic actions and reactions within the particle is not zero, *when the particle is being accelerated*. This last condition is very important. The EM self-force effect makes a clear distinction between constant velocity motion, when the self-force is always zero, as can be seen by inserting $\dot{x} = \text{constant}$ in (3), and accelerated motion, when it is unlikely ever to be zero.

This is very important because it saves Newton's first law, which proclaims that no external force should be required to keep an object moving at constant velocity. This law would not be true for any spatially extended charge distribution, not even one that is overall electrically neutral, unless the self-force were zero for constant velocity motion.

We can put this in a rather amusing way. When Newton's first law applies, i.e., there is no external force and hence we have constant velocity motion, Newton's third law also applies, in the above sense that the sum of all the EM actions and reactions within the particle is then zero. But when Newton's first law fails, i.e., we have an external force and accelerated motion, then Newton's third law also fails.

Another important point to mention is that the momentum-derived and self-force-derived masses in (2) and (4) are always exactly equal, i.e.,

$$m_{\text{EM}}^{\text{MDM}} = m_{\text{EM}}^{\text{SFDM}} \ .$$

This confirms the idea that momentum gets into the fields by our having to overcome this leading order term in the self-force.

We should say a word about the shape of the accelerating charge blob. As mentioned earlier, its shape in any given situation results from the equilibrium between the binding forces and the repulsive EM forces between charge elements making up the blob. When it changes its velocity, it will be desperately trying to FitzGerald contract to suit its instantaneous velocity, but with some delay effects depending on exactly how it is accelerated.

In self-force calculations, we generally assume rigidity of the blob, in the relativistic sense, known as Born rigidity. This basically amounts to assuming

that the charge blob always has exactly the same shape in its instantaneous rest frame, that is, in the instantaneously comoving inertial frame. Achieving this physically will depend as much on how the blob is accelerated as on the nature of the binding forces.

Note, however, that more sophisticated self-force calculations are possible that do not assume rigidity. The reader is referred to very recent work by Gralla, Harte, and Wald in Chicago [3].

We should also comment on the second term in the EM self-force, viz.,

$$F_{\text{self}}^{\text{rad}} := \frac{2}{3} \frac{e^2}{c^3} \dddot{x} \ . \tag{5}$$

This term does not depend on the spatial dimensions a of the system, and it does not even depend on its shape, as can be seen from the above expression. This remarkable term is the radiation reaction force. The rate of doing work against this part of the bootstrap force is exactly the rate of energy emission by radiation as given by the Larmor formula.

Note that we would lose this explanation for the EM radiation by accelerating electrons if we treated them as point particles, so this is another bonus of our earlier bootstrap hypothesis, according to which there are no point particles.

So we find that the EM self-force is in general a very complicated infinite sum of terms. Now obviously this expression would be vastly simplified if we could set a equal to zero. Then we would get rid of all the terms going as a, a^2, and so on, and we would still have the radiation reaction term, which we would like to keep since it explains why accelerating charged particles radiate electromagnetic energy. But as noted earlier, the problem with this strategy is that the leading order term goes to infinity as the spatial extent of the charge distribution tends to zero.

Then in 1938, Dirac came along [4], and he said, let us suppose that the electron acts on itself only through the second self-force term and not through the first or any of the higher order terms. And that peculiar suggestion is the basis of classical mass renormalization. Let us sketch briefly how that works.

We begin with Newton's second law, which says that force equals mass times acceleration. In this case the force is the sum of an external force causing the acceleration and the resulting EM self-force, and this is equal to the mass times the acceleration $\ddot{x}$, so we have

$$F_{\text{ext}} + F_{\text{self}} = m_{\text{bare}} \ddot{x} \ .$$

For the moment, let us call the mass the bare mass.

We now analyse the self-force into the infinite series, but drop all the terms going as a, a^2, and so on, since we intend to let a tend to zero at the end. We keep the radiation reaction term on the left of our new equation with the external force, but the clever thing here is that we group the potentially divergent leading order term with the bare mass times the acceleration. And the reason why this is useful is just that the leading order term is itself proportional to the acceleration. That is what makes this ploy work.

The result is a new version of Newton's second law, viz.,

$$F_{\text{ext}} + \frac{2}{3}\frac{e^2}{c^3}\dddot{x} = \left(m_{\text{bare}} + \alpha\frac{e^2}{ac^2}\right)\ddot{x} \ .$$

On the left we have the external force plus the radiation reaction force, and on the right a new mass term times the radiation. The renormalization step in the argument consists in saying that everything in the round brackets on the right-hand side must just be the measured mass. We do not worry about the fact that part of it must tend to infinity in the limit $a \to \infty$. We thus define the measured or renormalized mass to be

$$m_{\text{ren}} := m_{\text{bare}} + \alpha\frac{e^2}{ac^2} \ .$$

This is the process of classical mass renormalization.

Now the spherically symmetric charge shell is very nice because we can actually carry out the self-force calculation, but it is certainly not the simplest spatially extended charge distribution we can imagine. That must surely be the distribution shown in Fig. 1, i.e., two point charges A and B held some distance apart by some binding forces that we shall try not to think about.

How do we do a self-force calculation? We get the dumbbell moving in some arbitrary way in an inertial frame, describing the motion of each point charge, and then we use the remarkable Liénard–Wiechert formula which tells us the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ produced by each point charge and at every point in spacetime as a result of its specific motion. We can then calculate the fields created by A at B and vice versa, and hence the EM force exerted by A on B, and the EM force of B on A, and we simply add these together to estimate the total EM force the system exerts on itself.

Let us try to get an intuitive picture of how the self-force comes about by considering the charge dumbbell moving in some arbitrary way in one dimension, but perpendicular to its own axis, as shown in Fig. 2. Now B will produce some fields which will affect A slightly later due to retardation effects, and slightly later, A will have moved a little bit to the right. Now if A and B are like charges, they repel one another, so it looks as though the electric force of B on A will have a main component along the system axis AB and a small component in the direction of motion. Likewise it looks as though the electric force of A

Figure 1: Charge dumbbell, or toy model for the electron, at rest in an inertial frame, as used to investigate EM self-force. There are some binding forces and the system is in equilibrium under the forces between A and B
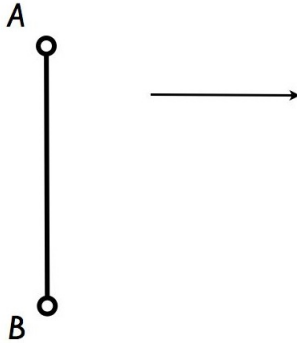
Figure 2: Charge dumbbell moving with arbitrary motion in one dimension in an inertial frame, with direction of motion perpendicular to its own axis

on B will have a main component along AB and a small component in the direction of motion. When we add these together, the components along the system axis will cancel exactly by symmetry, and it looks as though there will be two small components in the direction of motion that add up to give a net electric self-force in the direction of motion.

However, that could not possibly be correct. If it were, we could just get the charge dumbbell moving at constant velocity in 1D perpendicular to its axis and it would begin to accelerate itself! This would be very nice, of course, because it would solve all today's energy problems. However, this does not happen. Recall that the EM self-force is zero for constant velocity motion, so this intuitive picture breaks down.

This is in fact due to a remarkable result from Maxwell's theory of electromagnetism: when a charged particle moves with constant velocity, the electric field it produces is radial, not from its retarded position, but from its current position. It is as though the fields corrected themselves to first order for retardation effects, and first order is sufficient for constant velocity motion. Here we should think about Newton's first law, which would not be true for the charge dumbbell with this kind of motion if it were not for this fundamental result from electromagnetic theory.

So the intuitive picture is not valid and we are stuck with carrying out a somewhat tedious calculation using the Liénard–Wiechert formulas for the fields. What do we find for the charge dumbbell moving in an arbitrary way but perpendicular to its own axis as shown in Fig. 2? In fact, the electric force of A on B does indeed have a small component in the direction of motion and a main component along the system axis, and likewise for the electric force of B on A. However, the small components in the direction of motion are always counteraligned with the acceleration for like charges, and pay no heed to the direction of the velocity.

When we add together these electric forces, the axial components (along AB) do indeed cancel, while the components counteraligned with the acceleration for

like charges add up to produce a net electric self-force that is also opposed to the acceleration. The magnetic force of A on B lies along the system axis. Likewise for the magnetic force of B on A, and they cancel exactly.

Now here is an interesting thing. If the charges at A and B have opposite signs, then the self-force changes sign. So it will actually assist the acceleration! This may look somewhat suspicious, but it is exactly what one would expect for a charge dipole. Recall that the EM binding energy in a charge dipole is negative, and we expect any negative binding energy in a bound state particle to decrease its inertial mass.

So we expand the EM self-force in powers of d, the separation between A and B. We find that the Coulomb terms cancel so there is no term going as $1/d^2$. However, there is a residue going as $1/d$ and we find, as always, that the self-force diverges when d tends to zero. The EM self-force is in this case

$$F_{\mathrm{self}} = -\frac{e^2}{4c^2 d}\gamma(v)^3 \ddot{x} + O(d^0) \ . \tag{6}$$

Here we see that the factor α in Feynman's general self-force expression (3) is equal to $1/4$ for this shape of charge distribution with this motion.

Equation (6) is the result of a relativistic calculation and we see appearing the relativistic factor γ, function of the instantaneous speed. Note that it occurs as a cube, which may look awkward. However, this gamma factor is just right for renormalizing the relativistic version of Newton's second law, thanks to the simple identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\gamma(v)v\big] = \gamma(v)^3 \ddot{x} \ .$$

So here we have a dynamical explanation as to why the inertial mass of an object should increase with speed as $\gamma(v)$. It is because the EM self-forces increase in the appropriate way, in this case as a cube of γ, and this in turn is presumably a consequence of the Lorentz symmetry of Maxwell's equations. One would expect the same explanation to work for contributions to the inertial mass from other fields operating within a bound state particle, if those fields satisfy Lorentz symmetric equations.

So for this system, we obtain the renormalized mass

$$m_{\mathrm{ren}} := m_{\mathrm{bare}} + \frac{1}{4}\frac{e^2}{c^2 d} \ .$$

If we calculate the next term in the series expansion of the self-force, i.e., the one independent of d, we do indeed get the correct radiation reaction to explain the EM energy radiation by this charge distribution when it moves in this way.

We can also consider our charge dumbbell moving in an arbitrary way in one dimension, but this time along its own axis, as shown in Fig. 3. Here we must make some assumption about the length, such as a rigidity assumption, for example. In this case there are no magnetic forces of either A on B or B on A, and the leading order term in the electric self-force acts along the system axis. Once again it is always counteraligned with the acceleration for like charges A and B and pays no heed to the direction of the velocity.
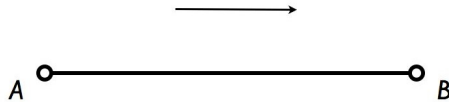
Figure 3: Charge dumbbell moving with arbitrary motion in one dimension in an inertial frame, with direction of motion along its own axis

The EM self-force is now

$$F_{\text{self}} = -\frac{e^2}{2c^2 d}\gamma(v)^3 \ddot{x} + O(d^0) \ . \tag{7}$$

Interestingly, the self-force-derived EM mass has changed. We now find

$$m_{\text{ren}} := m_{\text{bare}} + \frac{1}{2}\frac{e^2}{c^2 d} \ ,$$

so the factor α in Feynman's general self-force expression (3) is equal to $1/2$ for this shape of charge distribution with this motion. Note that we still get the right γ factor in (7) to be able to renormalize the relativistic version of Newton's second law.

So what we discover here is that the EM mass of an object depends on which way it moves relative to its own geometric configuration. In the present case we have

$$m_{\text{EM}}^{\text{SFDM}}(\text{longitudinal}) = 2m_{\text{EM}}^{\text{SFDM}}(\text{transverse}) \ .$$

This fact is hardly surprising when we consider what is going on within these bound state systems.

We can also consider the charge dumbbell rotating about a distant center as shown in Fig. 4. The distance R from the center of rotation should be considered much greater than the length d of the dumbbell. Now the acceleration is along the system axis and the velocity is perpendicular to it. But despite the very different configuration, we find once again that the leading order term in the self-force is counteraligned with the acceleration for like charges, i.e., it is radially outward from the center of rotation, and the relativistic factors work out perfectly to be able to renormalise the relativistic version of Newton's second law.

We can also consider the charge dumbbell rotating about a distant center as shown in Fig. 5. Once again the distance R from the center of rotation should be considered much greater than the length d of the dumbbell. This situation is quite different again. The acceleration is perpendicular to the system axis and the velocity lies along to it, but once again we find that the leading order term in the self-force is counteraligned with the acceleration for like charges, i.e., it is still radially outward from the center of rotation, and the relativistic factors work out perfectly to be able to renormalise the relativistic version of Newton's second law.
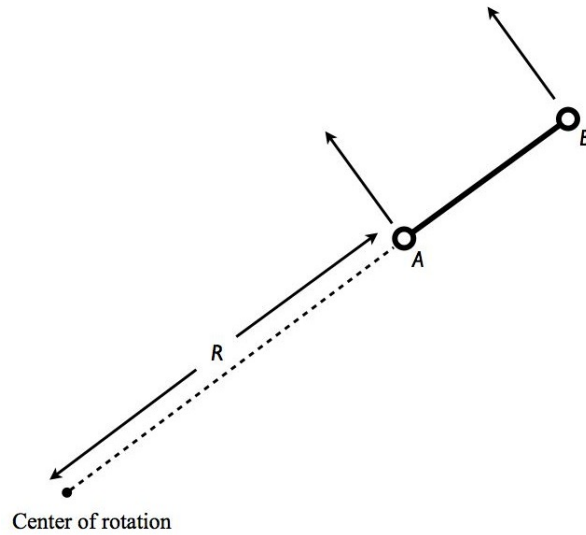
Figure 4: Charge dumbbell rotating at constant angular speed about a center at distance R from A, in such a way that the system axis AB always points to the center of rotation
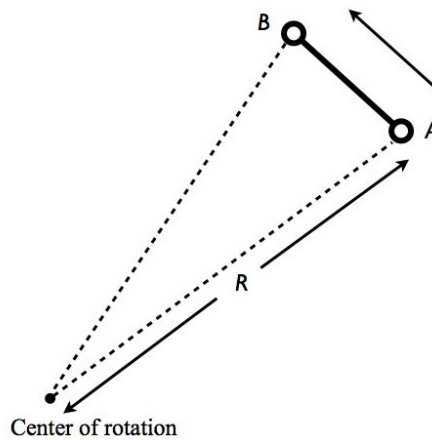

Figure 5: Charge dumbbell rotating at constant angular speed about a center at distance R from A, in such a way that the system axis AB is always perpendicular to the line joining its midpoint to the center of rotation

This raises a question: is the leading order term in the EM self-force always aligned or counteraligned with the acceleration? The answer is negative. Despite the positive results just mentioned for the charge dumbbell, it turns out that these are exceptions. When the charge dumbbell moves (rigidly) along an arbitrary worldline, the leading order term in the self-force contains a contribution along the system axis, in addition to the contribution aligned or counteraligned with the acceleration which can be removed by mass renormalization [5].

Note, however, that this unwanted term can be made to disappear by considering a spherically symmetric charge distribution. Indeed, the result for the charge dumbbell can be used in an integration to obtain the leading order term in the EM self-force for a spherically symmetric distribution, and one sees exactly how the unwanted contribution to this term drops out [5].

For uniform rigid acceleration of the charge dumbbell, Steane [8] has shown how the necessary binding forces in the system will themselves provide a self-force that exactly cancels the non-aligned part of the leading order term in the EM self-force, and this has been generalised to arbitrary rigid acceleration in [5]. The upshot is that the relativistic version of Newton's second law will be renormalisable in the point particle limit by a suitable adjustment of the inertial mass, provided we take into account the total self-force.

Furthermore, once again for arbitrary rigid acceleration of the charge dumbbell, Ori and Rosenthal [7] have shown how to derive an equation of motion without the need to involve the binding forces, and it is explained in [5] why this will be consistent with the other derivation in the point particle limit.

So what is the connection between all this and bound state particles? In modern particle physics, energy and mass are the same thing because

$$E = mc^2 .$$

So this is why we include binding energy in the inertial mass of composite particles, and everything is very simple. However, this hides another dynamical explanation, albeit a pre-quantum theoretical explanation. Binding forces in composite particles lead to bootstrap effects, and that is why binding energy must be included in their inertial mass. So the message here is that we include binding energies because they reflect the related self-forces in those bound states.

So what are the benefits of our radical bootstrap hypothesis, made at the beginning of this discussion. First of all, Newton's second law has become much simpler. It now says just $F = 0$. However, F has become much, much more complicated. It is the sum of the external force causing acceleration and all the resulting self-forces due to fields operating within the bound state particle:

$$F_{\text{ext}} + \sum_{\text{fields}} F_{\text{self}} = 0 . \tag{8}$$

The usual form for Newton's second law is obtained by analysing the self-forces and moving the leading order terms, proportional to the acceleration, to the right-hand side.

And now we deduce results that were simply imposed before. For example, we get a numerical value for the mass in the $F = ma$ form of the above law. And

since self-forces make a clear distinction between uniform velocity and changing velocity, we get a nice explanation of Newton's first law. We really understand why no external force is needed to keep a particle in constant velocity motion: it is because the particle then exerts no forces on itself.

At this point, one might wonder about the so-called Mach principle, which in one version suggests that a particle somehow (mysteriously, causally) gets its inertia from the overall distribution of matter and energy in the Universe. Under the present hypothesis, a particle gets its inertia solely and completely from within itself. What picks out the overall distribution of matter and energy in the Universe is that it seems to specify what passes for inertial frames, i.e., those frames in which our field theories of matter take on their simplest forms. But what seems more cogent would be the idea that the matter and energy in the Universe has evolved into its present distribution *because* of the existence of such frames, themselves a consequence of the underlying (Lorentz) symmetry of the field theories of matter.

Let us now look briefly at what happens in particle physics today. As mentioned before, we have several truly elementary particles like leptons and quarks which are not considered to have any internal structure, and then we have a whole host of bound state particles like baryons and mesons which we try to organise into multiplets.

What is a multiplet? Mathematically, it is a vector space carrying a representation of a group. But physically, it is just a set of particles with similar properties. And one of the properties that has to be similar across a multiplet is the inertial mass. So one task of modern particle physics is to group bound state particles together into sets with similar masses, and another is to explain why the masses are not quite equal across a given multiplet, something known in the jargon as mass splitting.

A good example is provided by the neutron, with inertial mass $939.5 \text{ MeV}/c^2$, and the proton, with inertial mass $938.2 \text{ MeV}/c^2$. These have very similar masses, so they are ideal for putting together in a multiplet. We thus make the hypothesis that their quantum states carry an irreducible representation of the isospin $\text{SU}(2)$ symmetry group:

$$|p\rangle = |T = 1/2, T_3 = 1/2\rangle , \qquad |n\rangle = |T = 1/2, T_3 = -1/2\rangle , \qquad (9)$$

characterized by an isospin value of $T = 1/2$.

Another example is the pion isotriplet. Once again, the positively charged, neutral, and negatively charged pions have very similar masses so they are ideal for grouping together into a multiplet, and we make the hypothesis that their quantum states carry an irreducible representation of the isospin $\text{SU}(2)$ symmetry group, but this time characterized by an isospin value of $T = 1$:

$$
\begin{aligned}
|\pi^+\rangle &= -|T = 1, T_3 = 1\rangle , & 139.6 \text{ MeV}/c^2 , \\
|\pi^0\rangle &= |T = 1, T_3 = 0\rangle , & 135.0 \text{ MeV}/c^2 , \\
|\pi^-\rangle &= |T = 1, T_3 = -1\rangle , & 139.6 \text{ MeV}/c^2 .
\end{aligned}
\qquad (10)
$$

14

Note the slightly smaller mass of the neutral pion, something we shall be able to explain shortly.

And of course we also have the quark flavour models for these particles. The neutron and proton are bound states of three quarks. The neutron is a bound state of one up quark and two down, while the proton is a bound state of one down and two up:

$$n = udd , \qquad p = uud .$$

The pions are mesons, i.e., quark–antiquark bound states:

$$\pi^- = d\bar{u} , \qquad \pi^0 = d\bar{d} , \ u\bar{u} , \qquad \pi^+ = u\bar{d} . \qquad (11)$$

So, for example, the positively charged pion is a bound state of an up quark and an antidown quark and the neutral pion a superposition of down–antidown and up–antiup. And note that the up and down quarks themselves form a multiplet. We make the hypothesis that their quantum states carry an irreducible representation of the isospin SU(2) symmetry group characterized by an isospin value of $T = 1/2$

Now these multiplets reflect a symmetry under the strong force. Recall that the strong interaction is at work in the above bound states, and indeed, they are largely held together by the strong interaction. In each case we model that by means of a strong interaction Hamiltonian H_{strong} which describes the energy of all the strong interactions going on within the bound state. Then the symmetry hypothesis is expressed by saying that this strong interaction Hamiltonian commutes with all the generators T_1, T_2, and T_3 of the isospin SU(2) symmetry group:

$$\left[H_{\text{strong}}, T_i \right] = 0 , \qquad i = 1, 2, 3 .$$

This in turn implies that the operator

$$S_{\text{strong}} = \exp \frac{\mathrm{i} H_{\text{strong}} t}{\hbar}$$

determining the time evolution of the bound state commutes with all generators and hence with all members of the isospin SU(2) group.

That is the mathematical statement of the symmetry assumption, but what does it mean physically? In fact it asserts that the strong interactions make no distinction between states in the multiplet. Put another way, the up quark and the down quark look the same as far as the strong force is concerned. However, electromagnetic interactions are also at work within these bound states, and these interactions do make a distinction here. The up quark has electric charge $+2/3$ and the down quark has electric charge $-1/3$, so these look very different to the EM interaction.

Now the mass of a state ψ in quantum theory is expressed as an expectation value of the relevant Hamiltonian in the given state, viz.,

$$M_\psi = \langle \psi | H | \psi \rangle , \qquad (12)$$

where H is the Hamiltonian modelling all the energy sources in the system. In this case then, we expect to have

$$H = H_{\text{strong}} + H_{\text{EM}} .$$

Inserting this in the expression (12) for the mass, we expect the masses of these bound states to be a sum of a strong contribution, which is expected to largely dominate, and a much smaller EM contribution:

$$
\begin{aligned}
M_\psi &= \langle\psi|H_{\text{strong}}|\psi\rangle + \langle\psi|H_{\text{EM}}|\psi\rangle \\
&= M_\psi^{\text{strong}} + M_\psi^{\text{EM}} .
\end{aligned}
$$

And the upshot of the symmetry hypothesis is that the strong contributions to the masses of all the bound states within a given multiplet will all be exactly equal, if the symmetry assumption holds exactly, while the much smaller EM contributions will differ from one state to the next.

The symmetry hypothesis concerning the strong interactions therefore explains the nearly equal masses of all the bound states within a given multiplet, while the EM contributions explain, at least in part, the mass differences. For example, we expect

$$m_{\text{n}} = m_{\text{nucleon}}^{\text{strong}} + m_{\text{n}}^{\text{EM}} , \qquad m_{\text{p}} = m_{\text{nucleon}}^{\text{strong}} + m_{\text{p}}^{\text{EM}} ,$$

so the strong contributions to the neutron and proton masses are expected to be exactly equal if the isospin symmetry holds exactly, while the EM contributions will explain, at least in part, the difference in mass between the two particles.

In modern particle physics then, the notion of electromagnetic mass is still there, but it is found in a very different way as the expectation value of the electromagnetic interaction Hamiltonian in the quantum state for the given bound state, e.g., for the proton and neutron,

$$m_{\text{p}}^{\text{EM}} = \langle\text{p}|H_{\text{EM}}|\text{p}\rangle , \qquad m_{\text{n}}^{\text{EM}} = \langle\text{n}|H_{\text{EM}}|\text{n}\rangle .$$

Before leaving the domain of particle physics, it is worth pointing out that the classical self-force model can be more sophisticated than just a charge shell or a charge dumbbell. This is fortunate because, as Feynman points out [2], if the proton were just a charged sphere and the neutron a neutral one, then from what was said earlier, we would expect the neutron to have the lower mass, and this is not the case [see (9)].

However, when we consider that the neutron and proton are each today considered to be bound states of three charged particles, there is absolutely no reason to think that, if we could actually carry out the classical self-force calculation, the neutron would turn out to have the lower EM mass simply on the grounds that it is overall electrically neutral.

And in fact the dumbbell model works rather well for the pions. As we can see from (11), the charged pions are composed of like charges, e.g., the positively charged pion $\pi^+ = u\bar{d}$ comprises an up quark with charge $+2/3$ and

an antidown with charge $+1/3$, while the neutral pion is composed of opposite charges. So from what was said earlier, we would expect the neutral pion to have the smaller inertial mass, and indeed it does [see (10)].

We can even use the crude classical dumbbell model to estimate the length of a pion, obtaining a value $d \sim 10^{-16}$ m, which accords quite well with estimates of pion diameters from cross-section measurements. Note that the spherical shell model also works surprisingly well here.

So the conclusion from all this is that mass splittings in multiplets of bound state particles are explained today, at least in part, by a quantum theoretical version of the classical self-force idea. But this comes with a warning. Mass splittings in multiplets of quark bound states are complicated by the different masses of the different quark flavours.

In fact, the isospin SU(2) flavour symmetry is broken by the fact that the up and down quarks have very different masses. Worse, we cannot measure these masses directly because it has so far proven impossible to isolate an individual quark, so we can only infer their masses from other observations, and a lot remains to be understood yet.

And what are the bonuses of our radical bootstrap hypothesis made at the beginning of this discussion, i.e., the assumption that there are no point particles and that the inertial mass of all particles results entirely from self-forces due to the various fundamental interactions operating within them? To begin with, we have a simpler version of Newton's second law, which now says just that the total force on any particle is always zero. And we have dynamical explanations for:

- Inertia and inertial mass, with the possibility of actually calculating the latter.

- The speed dependence of inertial mass as it is usually found in relativistic dynamics.

- The inclusion of binding energies in inertial mass.

- The EM energy radiation by accelerating charged particles.

But there is one more thing that we have not discussed in the above.

We have been talking about inertial mass, and it is well known that, according to very accurate measurements, the inertial mass of any object is exactly equal to its passive gravitational mass. Recall that the passive gravitational mass gauges the extent to which the particle is affected by gravity. Now it is sometimes said that general relativity explains why we should have this equality. The point is that any particle will follow a geodesic of the spacetime metric if there are no non-gravitational effects around to act on it. But that implies that any two particles, no matter what their inertial mass, will fall in the same way in the absence of any non-gravitational effects, i.e., free fall does not depend on the nature of the particle, and in particular on its inertial mass.

Actually, there are some provisos regarding this so-called geodesic principle, but they shall not concern us here. Let us just note that the above explanation

could be considered to turn things upside-down, since general relativity would not even be possible if it were not for the equality of inertial and passive gravitational mass, and it was the very discovery of this equality that led eventually to Einstein's formulation of the general theory of relativity (GR). Furthermore, there is a sense in which the above explanation lacks somewhat in impact, since it provides no mechanism.

If we place an object on our outstretched hand, we prevent it from free fall. In the GR picture there is only one force on the object, namely the force we exert upwards on it in order to push it off its geodesic. In the usual view of GR, gravity is not a force and there is no such thing as weight. This can be viewed as a linguistic adjustment, ensuring that forces are always associated with accelerations: the freely falling object has zero acceleration in the GR picture, while our supported object is being accelerated by the upward force from our hand.

However, we nevertheless feel the object pushing down on our hand, and it is interesting to wonder how it does that. One might say that this is just the reaction, according to Newton's third law, to the force we are exerting on the object. Indeed, by pushing on it, we slightly deform the microscopic structure of the object near its lower surface, and that structure will react to that. But there is another rather intriguing way of looking at this through the idea of the self-force. Let us examine how that works.

Just to set the scene in this GR view, consider a spacetime with coordinates

$$\left(y^0, y^1, y^2, y^3\right)$$

and a metric that only differs from the Minkowski form in the 00 component, which is the following function of one of the space coordinates:

$$g_{00} = \left(1 + \frac{gy^3}{c^2}\right)^2 \ ,$$

where c is the speed of light and g a constant with units of acceleration. These are supposed to be the coordinates one would set up in a laboratory held fixed relative to a distant gravitational source. The metric then describes a parallel gravitational field in the y^3 direction.

By the weak equivalence principle (WEP), at any event in any curved spacetime there is a neighbourhood with coordinates such that the metric looks Minkowskian to a good approximation. However, for this particular metric, the neighbourhood can be the whole spacetime, because this spacetime happens to be flat. The curvature is zero and there are no tidal effects.

The Minkowski coordinates whose existence is guaranteed by WEP are supposed to be the coordinates that would naturally be used by a freely falling observer, in the sense that an observer sitting at the space origin of such coordinates would be following a geodesic. The coordinates $\{y^\mu\}$ can be viewed as coordinates that might be adopted by a uniformly accelerating observer, in the sense that an observer sitting at the space origin of the $\{y^\mu\}$ coordinates would have uniform acceleration.

Now consider a charge shell held fixed at the origin of the $\{y^\mu\}$ system, i.e., a sphere supported against the uniform gravitational field. Since the four-acceleration is nonzero, this requires a force. The sphere is being pushed off its geodesic. As viewed from the freely falling frame, the sphere will appear to be accelerating (and it has nonzero acceleration according to the GR definition of acceleration).

We now import the theory of electromagnetism using the strong principle of equivalence (SEP). This states that any theory of non-gravitational physics will look roughly as it does in flat spacetime when described in locally inertial coordinates. The principle is usually formulated by taking the flat spacetime field equations for the non-gravitational effect and replacing all (inertial) coordinate derivatives by covariant derivatives (minimal extension of the theory from flat to curved spacetime).

According to SEP, an *exactly* equivalent view in this case (because our spacetime is in fact flat) is of the charge shell accelerating uniformly in a flat spacetime without gravity. But then we know that the sphere will exert an EM force on itself that opposes the acceleration, i.e., that acts toward the gravitational source. Put another way, the EM self-force will oppose the supporting force of the laboratory table upon which the sphere sits. In fact, it contributes to its weight.

But what is weight in GR? We said above that GR effectively does away with this notion. However, there is a natural way to reinstate it. We simply define it to be the negative of the force required to support the object at a fixed distance from the source:

$$W = -F_{\text{supp}} \ .$$

Then we need to reinstate passive gravitational mass, also rendered obsolete by general relativity. Since it is supposed to gauge the extent to which the object is affected by gravitational effects, the natural definition is to take it as (minus) the constant of proportionality between the weight and the four-acceleration A, i.e.,

$$W = -m_{\text{PG}} A \ ,$$

whence

$$F_{\text{supp}} = m_{\text{PG}} A \ .$$

We now propose the new dynamical law (8), viz.,

$$F_{\text{supp}} + F_{\text{self}} = 0 \ , \tag{13}$$

but this time in the GR framework. Here we are assuming the radical bootstrap hypothesis that all the inertial mass of an object arises due to the leading order terms in self-forces. Since

$$F_{\text{self}} = -m_{\text{inertial}} A + \text{smaller terms} \ ,$$

at least for spherically symmetric charge distributions, we deduce that

$$m_{\text{PG}} = m_{\text{inertial}} \ . \tag{14}$$

19

Note that all this is just standard theory, and precisely the way we always naturally think about things today, although with a different, and classical, mechanism. Think, for example, about the weight of a proton or atomic nucleus: we automatically include the binding energy as part of the weight. In GR, energy is gravitationally attractable.

Amusingly, if some part of the binding energy of a particle is negative, we have an antigravity effect. Think, for example, of a charge dipole lying on the laboratory table. The negative EM binding energy in the system will slightly decrease its weight. And we can see why no particle or object has ever been seen to float up into the air under the Earth's gravity! It is simply because no particle or object we have ever observed has ever been found to have negative *inertial* mass.

Of course, the above argument leading to (14) is circular. After all, we use this very same experimental result to formulate GR in the first place. But here perhaps we also have a mechanism, at least for bound state particles, or for the binding energy contributions to the inertial masses of bound state particles. Contrast with a naive special relativistic version of gravity in which gravity is just a force. The object supported by my outstretched hand is subject to two forces: the supporting force from my hand and its weight. But they exactly balance. There is no acceleration and there is no hope of an explanation of the kind just given.

In GR as it is usually presented, the supporting force is needed *because* the particle has nonzero four-acceleration. But we do not ask why a nonzero four-acceleration should require a supporting force, any more than we ask why an acceleration should require a force in Newtonian physics. The picture here, in the radical bootstrap hypothesis, is that all forces on an object must always exactly balance to give a zero resultant, as in (13).

And as mentioned earlier, self-forces make a clear distinction between uniform velocity and changing velocity. Here we understand why no supporting force is needed to keep a particle in free fall in GR. It is simply because it is not accelerating in the GR picture, so it does not exert any force on itself.

Acknowledgements

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